A short note on very large large cardinals
(without choice)

David Asperó∗

Abstract

I prove, in ZF together with the existence of a Reinhardt cardinal, that there is an ordinal \( \alpha \) with the property that there are rank–to–rank elementary embeddings sending \( \alpha \) arbitrarily high.

1 Introduction

By a classical result of Kunen ([2]), the natural limit of all currently known large cardinal axioms is inconsistent (in the ZFC context); specifically, under ZFC there is no nontrivial (i.e., different from the identity) elementary embedding from the universe \( (V) \) into itself.

This formulation of Kunen’s result is actually not very accurate: Given a nontrivial elementary embedding \( j : V_\lambda \rightarrow V_\lambda \) with \( \lambda \) a limit ordinal (a well studied situation not suspected to be inconsistent), \( V_\lambda \) is a model of ZFC, so it is consistent (modulo this large cardinal situation) that there is, from a very external point of view, a nontrivial elementary embedding \( j : V \rightarrow V \).

In fact, a much weaker assumption suffices for this: If \( \text{0}^\sharp \) exists, then there is (again externally from \( L \)), a nontrivial elementary embedding \( j : L \rightarrow L \). What Kunen’s proof shows is that there is no nontrivial elementary embedding \( j \) from \( V \) into itself such that \( j \) is “within sufficient reach from \( V \).” One way to make precise sense of the phrase “\( j \) is within sufficient reach from \( V \)” (and the way I shall be considering in this note) is to say that the structure \( V = \langle V, \in, j \rangle \) satisfies the Axiom scheme of Replacement for formulas in the language of \( V \) (see for example [1]).

∗Universidad Nacional de Colombia (Bogotá, Colombia)

1A weaker way of making sense of it is to say that \( j \) is first order definable over \( \langle V, \in \rangle \).
Introduction

is that if \( \langle V, \in \rangle \models \text{ZFC} \) and \( j \) is a nontrivial elementary embedding from \( V \) into itself with respect to formulas in the \( \{\in\} \)-language, then \( \mathcal{V} = \langle V, \in, j \rangle \) does not satisfy Replacement in the language of \( \mathcal{V} \).

The use of the Axiom of Choice features prominently in Kunen’s original proof, as well as in all alternative proofs of the inconsistency. In fact, four decades after [2], it is still widely open whether the Axiom of Choice is needed to derive a contradiction from the assumption that there is a nontrivial elementary embedding from \( V \) into itself; more precisely, it is open whether the following theory \( R_{ZF} \) is consistent.\(^2\)

**Definition 1.1** Let \( \mathcal{L} \) be the language \( \{\in, j\} \), where both \( \in \) and \( j \) are binary relation symbols. \( R_{ZF} \) consists of all ZF axioms (in the \( \{\in\} \)-language), together with the scheme of sentences expressing that \( j \) is a nontrivial elementary embedding from \( \langle V, \in \rangle \) into \( \langle V, \in \rangle \), and together with the Axiom scheme of Replacement for \( \mathcal{L} \)-formulas.

\( R_{ZF} \) is an extremely strong theory. For example, by an unpublished result of Woodin, in \( R_{ZF} \) one can build models of ZFC realizing all large cardinal axioms not known to be inconsistent (in the ZFC context).

2 Obtaining many rank–to–rank embeddings from \( R_{ZF} \)

**Theorem 2.1** Let \( \langle V, \in, j \rangle \) satisfy \( R_{ZF} \). Let \( \kappa_0 = \text{crit}(j) \) and let \( \lambda = \sup_n \kappa_n \), where \( \kappa_{n+1} = j(\kappa_n) \) for all \( n \). There is a cardinal \( \pi < \kappa_0 \) and an ordinal \( \alpha \) such that for every ordinal \( \beta \) there is some ordinal \( \mu > \lambda \) and some \( i \) such that

1. \( i : V_\mu \rightarrow V_\mu \) is an elementary embedding with critical point \( \pi \),
2. \( i(\lambda) = \lambda \), and
3. \( i(\alpha) > \beta \)

**Proof:** Given an ordinal \( \gamma \), let \( \kappa^\gamma \) be the least cardinal \( \kappa \) for which there is an ordinal \( \mu \) and an elementary embedding \( i : V_\mu \rightarrow V_\mu \) with \( \text{crit}(i) = \kappa \),

\(^2\)R stand for Reinhardt.
$i(\lambda) = \lambda$, and such that $\{\xi \in \mu : \lambda \leq \xi, i(\xi) = \xi\}$ has order type $\gamma$ (if there is such a cardinal).

Note that suitable fragments of $j$ witness that $\kappa^\gamma$ is defined for every $\gamma$. Moreover, $\alpha < \beta$ implies $\kappa^\alpha \leq \kappa^\beta \leq \text{crit}(j)$ ($\kappa^\alpha \leq \kappa^\beta$ is immediate, and $\kappa^\beta \leq \text{crit}(j)$ is, again, witnessed by a suitable restriction of $j$). Hence, there is some ordinal $\alpha$ and some $\pi \leq \kappa$ such that $\kappa^\beta = \pi$ for every ordinal $\beta \geq \alpha$. Note that in fact $\pi < \text{crit}(j)$ since $\kappa$ is definable in $(V, \in)$.

Now pick any limit ordinal $\beta > \alpha$ and any embedding $i : V_\mu \rightarrow V_\mu$ witnessing $\kappa = \kappa^\beta$. All we need to do is check that $i(\alpha) \geq \beta$.

**Claim 2.1.1** *For every $\gamma \in [\alpha, \beta)$, $V_\mu \models \kappa^\gamma = \pi$.***

**Proof:** Let $\delta$ be the $\gamma$-th member of the strictly increasing enumeration of $\{\xi < \mu : \lambda \leq \xi, i(\xi) = \xi\}$. Then $i \upharpoonright V_{\delta+1} : V_{\delta+1} \rightarrow V_{\delta+1}$ can be naturally coded as a set belonging to $V_{\delta+2} \subseteq V_\mu$ ($V_{\delta+2} \subseteq V_\mu$ is true since $\{\xi < \mu : \lambda \leq \xi, i(\xi) = \xi\}$ has order type $\beta$, which is a limit ordinal above $\gamma$). It follows that $V_\mu$ thinks that $\kappa^\gamma$ exists and that $\kappa^\gamma \leq \pi$ (as witnessed by $i \upharpoonright V_{\delta+1}$). On the other hand, $V_\mu \models \kappa^\gamma < \pi$ is clearly impossible since $\gamma \geq \alpha$. $\Box$

From the above claim it follows that if $i(\alpha) < \beta$, then $V_\mu \models \kappa^{i(\alpha)} = \pi$. But also $V_\mu \models \kappa^{i(\alpha)} = i(\kappa^\alpha) = i(\pi) > \pi$ by elementarity of $i$ and since $\pi = \text{crit}(i)$. This contradiction shows $i(\alpha) \geq \beta$. $\Box$

**References**
