State Dependent Utility

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Abstract
We propose a new approach to utilities in (state) complete markets that is consistent with state-dependent utilities. Full solutions of the optimal consumption and portfolio problem are obtained in a very general setting which includes several functional forms for utilities used in the current literature, and consider general restrictions on allowable wealths. As a secondary result we obtain a suitable representation for straightforward numerical computations of the optimal consumption and investment strategies. In our model utilities reflect the level of consumption satisfaction of flows of cash in future times as they are (uniquely) valued by the market when the economic agents are making their consumption and investment decisions. The theoretical framework used for the model is one proposed by the author in Londoño, Jaime A. (2008) “A more general valuation and arbitrage theory for Itô processes”, Stoch. Anal. Appl., 26:4, 809–831. We develop the martingale methodology for the solution of the problem of optimal consumption and investment in this setting.

1 Introduction
The problem of optimal consumption and investment for a “small investor” whose actions do not influence market prices is at the core of portfolio management and it is the building block for the development of equilibrium theory. The modern treatment of this problem when asset prices follows Itô processes started with the seminal works of Merton [39] and Merton [40]. Using a “martingale” approach, Cox and Huang [10], and Karatzas et al. [26] solved the problem in more general settings in the case of complete markets. A representation formula is derived in Ocone and Karatzas [41] in terms of expectation of random variables which involve Malliavin derivatives of the coefficients of the model. The latter gives theoretical formulas for optimal portfolios and consumption strategies.

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In order to obtain numerical representations for the structure of optimal portfolios and consumption processes it is natural to use methods based on the dynamic-programming approach. However numerical schemes based on PDEs become increasingly difficult to evaluate when the dimension of the underlying state variable increases and even standard techniques are somehow inappropriate for the solution of the PDE that arises in small dimensions (Dangl and Wirl [13]). As a result attention has been directed to models admitting closed form solutions (i.e. Watchter [43], Kim and Omberg [30], Lioui and Poncet [33]), specifications which are computationally tractable based on dynamic programming techniques (Brennan et al. [3], Brennan [2], Brennan and Xia [4], Campbell et al. [5]), discrete time models based on approximated Euler equations (Balduzzi and Lynch [1], Dammon et al. [12], Campbell and Viceira [6], and Campbell and Viceira [7]) or Monte Carlo techniques (Cvitanić et al. [11] and Detemple et al. [15]).

However, the main drawback of the standard models for optimal consumption and investment is their lack of agreement with empirical data. These inconsistencies are documented with the name of several puzzles such as the “equity premium puzzle” (Mehra and Prescott [37]), the “risk-free rate puzzle” (Weil [44]), and “risk-aversion puzzle” (Jackwerth [25]). In order to address these problems several generalizations have been suggested. One of these proposes habit formation for consumers. Some examples are Constantinides [9], Hindy and Huang [21], Hindy et al. [22], Hindya et al. [23]. Another approach is the construction of recursive utility. Some references for this are Duffie and Epstein [16], Epstein and Zin [17] and Lazrak and Quenez [32]. A different way to account for the discrepancy of theory and empirical data is the assumption of transaction costs for changes in consumption levels. Some references for this approach are Magill and Constantinides [36], Shreve and Soner [42], Davis and Norman [14] to cite a few.

In fact one of the reasons why the standard utility models fail to fit economic behavior might be the fact that state independent utilities are not appropriate for modeling the behavior of human beings. For instance, see Karni [29], Karni [28]. Partly motivated by the above, some literature in finance has focused on state-dependent utilities to explain the behavior of individual consumers and investors. Some recent references are Chabi-Yo et al. [8], Melino and Yang [38], Gordon and St-Amour [18], and Gordon and St-Amour [19] among others. For instance, Melino and Yang [38] generalize the model of Epstein and Zin [17] by allowing preferences where the elasticity of inter-temporal substitution and coefficient of risk aversion are state dependent. In a Mehra-Prescott economy they are able to match the US historical first two moments of the returns on equity and the risk-free rate. Similarly, Gordon and St-Amour [18, 19] using expected-utility preferences are able to find a good match to post WWII data.

In this paper we propose a new approach for utilities in state complete markets. See Londoño [34] and Londoño [35] for the definition of a state complete market. Mathematically this model looks similar to the standard model for utilities, but its interpretation is consistent with a model with state-dependent utilities. The traditional approach is to consider that utilities reflect the level of “happiness” for consumption levels in the future (discounted by the value of money in a bank account). See Karatzas and Shreve [27]. However our guiding principle in this paper is different: we believe that agents have utilities
for consumption of flows of money in future times as they are valued (by the market) at the time when they are making their consumption and investment decisions. Another way to look at this, is that people tend to value things according to their social and economic context, instead of just looking at quantitative values. For instance people tend to appreciate more the ability to have enough money to pay off their debts in depression times than the ability to buy luxuries in good times. In fact, state dependence is consistent with recent evidence from experimental psychology, where good times bring a positive mood for investors and a heightened pain from any potential loss (see Isen [24] and references therein). A detailed account that explains why state-dependent utilities have a potential to resolve the equity premium puzzle, is discussed in Melino and Yang [38]. In a Mehra-Prescott economy, they show that in order to match data on the first two moments of asset returns is necessary a stochastic discount factor that is very sensitive to the current state, and that it should be consistent with a counter-cyclical pattern of risk aversion. However, with iso-elastic expected utility preferences, the stochastic discount factor varies over time only with the rate of consumption growth realized next period.

The above remarks change completely the optimization problem to consider, with the advantage that most of the tools used to solve the old problem can be used in this setting. In particular the martingale methodology is available. A consequence of adopting this approach is that full solutions of the optimal consumption and portfolio problem are obtained in a very general setting that includes several of the functional forms for utilities in the literature, and considers quite general restrictions on allowable wealths. As a secondary result we obtain suitable representation for straightforward numerical computations of the optimal consumption and investment strategies.

The theoretical framework used to solve the above problem is the one proposed in Londoño [35]. In this section we just named the processes described in the cited paper as consistent measurable processes; these are processes whose evolution between any two times only depends on the evolution of the underlying Brownian motion and satisfies some consistency conditions. The methodology described can be used in processes that are a generalization of Brownian flows (Kunita [31]), and is applicable to those processes that are the solutions of classical Itô stochastic differential equations, even when the volatilities and drifts are just locally δ-Hölder continuous for some δ > 0. It can also be straightforwardly adapted to stochastic volatility models whose evolution of the underlying drifts and volatilities are described by classical stochastic Itô differential equations. As discussed in Londoño [35] the theoretical framework is potentially useful when the underlying randomness is generated by a (not necessarily continuous) Lévy process. At the same time the theoretical framework described in Londoño [35] is a particular case of the theory of arbitrage and valuation presented in Londoño [34]. To the best of our knowledge this theory of arbitrage and valuation is the most general existing theory in the case of (continuous) semi-martingales driven by Brownian filtrations with continuous coefficients.

Next we describe the contents of the paper. In section 2 we review the model and definitions presented in Londoño [35], and review the definitions of utility that we use in this paper. In section 3 we present a martingale methodology needed to address the cited problem for the model described in Londoño [35], that plays the role of the martingale methodology of Cox and Huang [10],
Karatzas et al. [26] and Ocone and Karatzas [41] in the current context. Finally, in section 4 we present the main results on optimal consumption and investment.

2 The model

First we introduce some notation which will be frequently used in this paper. Let $D \subset \mathbb{R}^k$ be a open connected set. Let $m$ be a non-negative integer. We denote by $C^{m,\delta}(D; \mathbb{R}^n)$ the Fréchet space of $m$-times continuous differentiable functions whose $m$-order derivatives are $\delta$-Hölder continuous with seminorms $\|f\|_{m,\delta; K}$ defined in Kunita [31], Section 3.1 where $K \subset D$ is a compact set and $0 \leq \delta \leq 1$. In case $m = 0$ (or $\delta = 0$) we denote $C^{m,\delta}(D; \mathbb{R}^n)$ simply by $C^0(D; \mathbb{R}^n)$ ($C^m(D; \mathbb{R}^n)$).

We assume a $d$-dimensional Brownian Motion $\{W(t), \mathcal{F}_t; 0 \leq t \leq T\}$ starting at 0 and defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F} = \mathcal{F}_T$ and $\{\mathcal{F}_t, 0 \leq t \leq T\}$ is the $\mathbb{P}$ augmentation by the null sets of the natural filtration $\mathcal{F}^W_t = \sigma(W(s), 0 \leq s \leq t)$. Let $(\mathcal{F}_{s,t}) = \{\mathcal{F}_{s,t}, 0 \leq s \leq t \leq T\}$ be the two parameter filtration where $\mathcal{F}_{s,t}$ is the smallest sub-\(\sigma\)-field containing all null sets and $\sigma(W_u | s \leq u \leq t)$, where $W_u \equiv W(u) - W(s)$. For each $0 \leq s \leq T$ we also define the $\sigma$-field $\mathcal{P}_s$ of progressively measurable sets after time $s$ as the $\sigma$-field of sets $P \in \mathcal{B}([s, T]) \otimes \mathcal{F}_{s,T}$, the product $\sigma$-field, such that $\chi_P(t, \omega), t \geq s$, is a $\mathcal{F}_{s,t}$ progressively measurable (in $t$) process, where $\chi$ is the indicator function. We denote by $\mu_s$ the measure on $\mathcal{P}_s$ defined by $\mu_s(P) = \mathbb{E} \int_s^T \chi_P(t, \omega) \, dt$.

We assume $n + 1$ stocks whose evolution price process $P$ is a consistent semi-martingale of class $C(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$. A detailed account of consistent processes, financial markets, wealth evolution structure and other concepts used in this chapter can be found in Londoño [35]. For $0 \leq i \leq n$ we define the price per-share process for the $i$-stock, $P_i$, to be the $P$-consistent semi-martingale process of class $C(\mathbb{R}^{n+1}; \mathbb{R}^+)$, $P_i = \{P_i(s, t, p) = \pi_i \circ P(s, t, p), p \in \mathbb{R}^{n+1}, 0 \leq s \leq t \leq T\}$ where $\pi_i$ denotes the projection on the $i$-component. We assume $P$-consistent semi-martingale processes $\sigma_{i,j}, b_i, \delta_i, r_i$, and $\theta_i$ of class $C^\delta(\mathbb{R}^{n+1}; \mathbb{R})$ for some $\delta > 0$, where $\mathbb{R}_+$ denotes the set of real positive numbers. It is assumed that $\sigma_{i,j}, b_i, \delta_i, r_i$, and $\theta_i$ relate to $P$ through the following stochastic differential equations

$$dP_i(s, t, p) = P_i(s, t, p) \left[ b_i(s, t, p) dt + \sum_{1 \leq j \leq d} \sigma_{i,j}(s, t, p) \, dW^j_s(t) \right]$$

where $W^j_s(t) = W^j(t) - W^j(s)$, and,

$$dP_0(s, t, p) = P_0(s, t, p) \left[ (r(s, t, p) + \|\theta(s, t, p)\|^2) \, dt + \sum_{1 \leq j \leq d} \theta_j(s, t, p) \, dW^j_s(t) \right]$$

where $\| \cdot \|$ denotes the Euclidean norm.
Throughout this paper we shall assume that \( \theta(s,t,p) \in \ker^{-1}(\sigma(s,t,p)) \) for all \( t \), (where \( \ker^{-1}(\sigma(s,t,p)) \) denotes the orthogonal complement of the kernel of \( \sigma(s,t,p) \)) and,

\[
b(s,t,p) + \delta(s,t,p) - r(s,t,p)1_n = \sigma(s,t,p)\theta(s,t,p)
\]

a.e. \( \mu_x \), for all \( p \in \mathbb{R}^{n+1}_+ \), and \( 0 \leq s \leq T \), where \( 1_n^* = (1, \ldots, 1) \in \mathbb{R}^n \). This latter assumption is equivalent to the non existence of state-tame arbitrage opportunities (see Londoño [34]). We point out that equation (2) is different from the equation in Londoño [35]. In fact this choice is made because in state-complete markets \( P_0 \) would be a hedgeable financial instrument (Londoño [34]).

The process of bounded variation \( B = \{B(s,t,p)\} \), whose evolution \( B(s,\cdot,p), p \in \mathbb{R}^{n+1}_+ \), \( 0 \leq s \leq T \) is given by the solution of stochastic differential equation

\[
dB(s,t,p) = B(s,t,p)r(s,t,p)dt, \quad B(s,s,p) = 1, \text{ for } 0 \leq s \leq t \leq T
\]

will be called the bond price process.

We shall say that \( \mathcal{M} = (P,b,\sigma,\delta,r,p^0) \) is a financial market with terminal time \( T \) and initial time \( 0 \), if \( b = (b_1, \ldots, b_n) \) is a vector of rate of return processes, \( \sigma = (\sigma_{ij}) \) is a matrix of volatility coefficient processes, \( \delta = (\delta_1, \ldots, \delta_n) \) is a vector of dividend rate processes, \( r \) is an interest rate process, and \( \theta'(s,t,p) = (\theta_1(s,t), \ldots, \theta_n(s,t)) \) is the market price of risk, and \( p^0 \in \mathbb{R}^{n+1}_+ \) is a vector of initial prices.

We define the state price density process to be the continuous \( C(\mathbb{R}^{n+1}_+; \mathbb{R}_+) \) semi-martingale process defined by

\[
H(s,t,p) = B^{-1}(s,t,p)Z(s,t,p) \quad \text{for } p \in \mathbb{R}^{n+1}_+, 0 \leq s \leq t \leq T
\]

where

\[
Z(s,t,p) = \exp \left\{ - \int_s^t \theta'(s,u,p) \, dW_s(u) - \frac{1}{2} \int_s^t \| \theta(s,u,p) \|^2 \, du \right\}
\]

for \( 0 \leq s \leq t \leq T \), and \( B^{-1}(s,t,p) = 1/B(s,t,p) \).

In this paper, we assume that \( \sigma(s,t,p) \) is a matrix of maximal rank. In this case the financial market \( \mathcal{M} \) is state-complete (see Londoño [35], Londoño [34]). It follows that if there exist an “equivalent martingale measure” this is unique (see Harrison and Pliska [20]) and \( H(0,t,p) \) is the unique “state price density process”.

Next we discuss the concept of utility that we shall use in this paper.

**Definition 1.** Consider a function \( U: (0, \infty) \rightarrow \mathbb{R} \) continuous, strictly increasing, strictly concave and continuous differentiable, with \( U'(\infty) = \lim_{x \rightarrow -\infty} U'(x) = 0 \) and \( U'(0^+) \leq \lim_{x \rightarrow 0^+} U'(x) = \infty \). Such a function will be called a utility function.

Classic examples of utility functions are \( U_\alpha(x) = x^\alpha / \alpha \) for some \( \alpha \in (0,1) \), \( 0 \leq x < \infty \), and \( U(x) = \log(x) \). For every utility function \( U(\cdot) \), we shall denote by \( I(\cdot) \) the inverse of the derivative \( U'(\cdot) \); the functions \( I(\cdot) \), and \( U'(\cdot) \), are strictly decreasing and map \( (0, \infty) \) onto itself with \( I(0^+) = U'(0^+) = \lim_{x \rightarrow 0^+} U'(x) = \infty \), \( I(\infty) = \lim_{x \rightarrow \infty} I(x) = U'(\infty) = 0 \). We extend \( U \) by \( U(0) = U(0^+) \), and we keep the same notation to this extension.
it would be clear to the reader to which function we are referring. It is a well
known result that
\[
\max_{0 < x < \infty} (U(x) - xy) = U(I(y)) - yI(y), \quad 0 < y < \infty
\]

**Definition 2.** Consider a continuous function \( U_1 : [0, T] \times (0, \infty) \to \mathbb{R} \), such that
\( U_1(t, \cdot) \) is a utility function in the sense of Definition 1 for all \( t \in [0, T] \). It follows
that \( I_1(t, x) \triangleq (\partial U_1(t, x)/\partial x)^{-1} \), the inverse of the derivative of \( U \), is a continuous
function. Similarly if a utility function \( U_2 : (0, \infty) \to \mathbb{R} \) is given then \( I_2(x) \triangleq
(\partial U_2(x)/\partial x)^{-1} \) is continuous. Let us denote
\[
\mathcal{X}(t, y) \equiv I_2(y) + \int_t^T I_1(t', y) \, dt'.
\]

We shall call a couple of functions as above a state preference structure.

Under the conditions outlined in the previous definition, it is easy to see
that \( \mathcal{X} : [0, T] \times (0, \infty) \to (0, \infty) \) is a continuous function with the property that
for each \( t \), \( \mathcal{X}(t, \cdot) \) maps \( (0, \infty) \) onto itself, is strictly decreasing with \( \mathcal{X}(t, 0^+) = \lim_{y \downarrow 0} \mathcal{X}(t, y) = \infty \)
and \( \mathcal{X}(t, \infty) = \lim_{y \rightarrow \infty} \mathcal{X}(t, y) = 0 \).

We extend \( U_1 \) and \( U_2 \) by defining \( U_1(t, 0) = U(t, 0^+) \), for all \( 0 \leq t \leq T \)
and \( U_2(0) = U_2(0^+) \), and we keep the same notation to the extension of \( U_1 \) to
\( [0, T] \times [0, \infty) \), and the extension of \( U_2 \) to \([0, \infty) \). We hope that it would be clear
to the reader to which function we are referring.

We point out that \( \mathcal{X}^{-1} \) defined for each \( t \) as \( \mathcal{X}^{-1}(t, \cdot) \), the inverse of \( \mathcal{X}(t, \cdot) \),
share the same mentioned properties of \( \mathcal{X} \). We next discuss the meaning of those
utility functions defined above. We should interpret \( U_1(t, x) \), for \( t \in [0, T] \) the
level of “happiness” for an agent consuming \( x \) units of wealth per unit of time
at time \( t \), as valued by the (state complete) market at time \( 0 \), when the agent
is planning its consumption. Similarly, we should understand for \( U_2(x) \) the
level of “happiness” for an agent having a final wealth of \( x \) units (at time \( T \))
as valued at time \( 0 \) by the (state complete) market. See the discussion after the
definitions of the problems of optimization in section 4. This is contrary with
the traditional approach where an agent has preferences on their consumption behavior according to their value as discounted by a bank account, and is closer in
approach to a utility function that is state dependent. See the literature on
state dependent utilities cited above.

For \( s \leq t \) define \( \alpha(s, t) = \mathcal{X}(s, \mathcal{X}^{-1}(t, \cdot)) \). Then \( \alpha(s, t) = \alpha(s, t') \circ \alpha(t', t) \)
for all \( s, t, \) and \( t' \) in \([0, T] \), where \( \circ \) denotes standard composition of functions. We also observe that if \( \alpha'(s, t) \equiv I_1(s, \mathcal{X}^{-1}(t, \cdot)) \) then \( \alpha'(s, t) \circ \alpha(t, s) = \alpha'(s, s) \).
Throughout this paper we shall assume the following condition on the utility structure.

**Condition 1 (Homogeneity).** Let \( (U_1, U_2) \) be a state preference structure defined as
above. For all \( s, t \in [0, T] \) there exist constants \( \alpha_{s,t} \) and \( \alpha'_s \) such that \( \alpha(s, t)(x) = \alpha_{s,t} x \), and \( \alpha'(s, s)(x) = \alpha'_s x \) where \( \alpha(s, t) \) and \( \alpha'(s, s) \) are defined as the previous paragraph. In this case we say that \((U_1, U_2) \) is a homogeneous state preference structure.

A way to see this is to say that the structure for the utility preferences remains the same as time evolves. We next describe some important examples
that fit the previous conditions.
Example 1. Assume a continuous positive function \( h : [0, T] \to (0, \infty) \), and assume that \( U_1(t,x) = x^\alpha h(t) \) and \( U_2(x) = cx^\alpha \) with \( \alpha \in (0,1) \) and \( c \geq 0 \). This is an state preference structure that satisfies Condition 1. Indeed in this case
\[
\alpha_{s,t} = \frac{c^{1/(1-\alpha)} + \int_s^t h^{1/(1-\alpha)}(t') \, dt'}{c^{1/(1-\alpha)} + \int_s^t h^{1/(1-\alpha)}(t') \, dt'}, \quad \alpha_t = \frac{h^{1/(1-\alpha)}(t)}{c^{1/(1-\alpha)} + \int_s^t h^{1/(1-\alpha)}(t') \, dt'}
\]

Example 2. Assume a continuous positive function \( h \) as above, and assume that \( U_1(t,x) = h(t) \log(x) \) and \( U_2(x) = c \log(x) \), with \( c > 0 \). It follows that this is a state preference structure that satisfies Condition 1 with
\[
\alpha_{s,t} = \frac{c + \int_s^t h(t') \, dt'}{c + \int_s^t h(t') \, dt'}, \quad \alpha_t = \frac{h(t)}{c + \int_s^t h(t') \, dt'}
\]

Example 3. Let \( U_1(t,x) = h(t)u(x/h(t)) \) and \( U_2(x) = cu(x/c) \), where \( u(\cdot) \) is a utility function, \( h(\cdot) \) is a positive continuous function and \( c > 0 \). It follows that \((U_1, U_2)\) is a state preference structure that satisfies Condition 1. In this case
\[
\alpha_{s,t} = \frac{c + \int_s^t h(t') \, dt'}{c + \int_s^t h(t') \, dt'}, \quad \alpha_t = \frac{h(t)}{c + \int_s^t h(t') \, dt'}
\]

In particular, for any given \( c > 0 \) \( U_1(t,x) = u(x) \), and \( U_2(x) = cu(c^{-1}x) \) define a state preference structure that satisfies Condition 1.

Let us point out that the coefficients \( \alpha_{s,t} \) and \( \alpha_t \), for all \( s, t \) in Example 3 are independent of the function \( u \), and are identical to the coefficients obtained for the similar coefficients in example 2. In Theorems 2, 3, and 4 below we show that for an agent that consume and invest optimally as explained in the problems 1, 2, and 3. respectively, the values of these variables depend on the utility functions only through the coefficients \( \alpha_{s,t} \) and \( \alpha_t \).

3 A Martingale approach

We remind the reader that we assume throughout this paper that equation (3) holds, and that \( \sigma(s,t,p) \) is a matrix with maximal rank for all \( s, t, p \). It follows that if there exist an “equivalent martingale measure” this is unique (see Harrison and Pliska [20]). Therefore under the assumptions in this paper, if there exist an “equivalent martingale measure”, then \( H(0,t,p) \) is the unique “state price density process”. However, the theory developed in Londoño [34], and Londoño [35] allows to hedge and to price any reasonable financial instrument even in the case where \( \mathbb{E}Z(s,T,p) < 1 \) for some \( s \) and \( p \). Londoño [34] and references cited therein discuss examples of this kind of models with a clear economic meaning.

Next, we define a refinement of the concept of wealth-income structure introduced in Londoño [35], that is needed to formulate the optimization problems that we are concerned.

Definition 3. Assume that \((X, \Gamma)\) is a wealth-income evolution structure (See Londoño [35] for this definition.). Assume that \( \Gamma \equiv E - C \), where
\[
dC(s,t,x,p) = c(s,t,x,p) \, dt \quad (7)
\]
and,

$$dE(s, t, p) = \varepsilon(s, t, p) \, dt$$  \hspace{1cm} (8)$$

for non-negative \((X, P)\) consistent semi-martingale processes \(c\) and \(\varepsilon\) of class \(C^\delta\) for some \(\delta > 0\). Moreover assume that

$$\mathbb{E} \left[ \int_s^T H(s, u, p) \varepsilon(s, u, p) \, du \right] < \infty$$

for all \(p \in \mathbb{R}^n_+\), and \(0 \leq s \leq T\), where as before \(H(s, t, p)\) is the state price density process. Similarly assume that

$$\mathbb{E} \left[ \int_s^T H(s, u, p) c(s, u, x, p) \, du \right] < \infty$$

for all \(p \in \mathbb{R}^n_+, x \in \mathbb{R}\) and \(0 \leq s \leq T\).

We should say \((X, c, \varepsilon)\) as above is a rate of consumption and endowment evolution structure. We shall say that \(c\) is the consumption rate evolution structure, and \(\varepsilon\) is the endowment rate evolution structure. We also say that \(E\) is a cumulative endowment structure, and \(C\) is a cumulative consumption structure.

A subsistence random field \(L\) is a \(P\) consistent process with drift and diffusion of class \(C^\delta\) for some \(\delta > 0\) where \(L(s, \cdot, p)H(s, \cdot, p)\) is uniformly bounded below for all \(p, s, t\) (where the bound might depend on \(p\) and \(s\)) such that

$$\mathbb{E} \left[ H(s, t, p)L(s, t, p) \right] < \infty$$

for all \(p, s\) and \(t\).

It is natural to believe that the evolution of income due to labor only depends on the evolution of the state of the economy and not on the current wealth of an agent.

Typically we are interested in consumption and endowment evolution structures whose wealth remains above some given process. Next we present the definition that embodies this idea.

**Definition 4.** Let \((X, \varepsilon, c)\) be a hedgeable (by a state tame portfolio) rate of consumption and endowment evolution structure, as in Definition 3, with portfolio evolution structure \((\pi_0, \pi)\). We shall say that the couple \((\pi, c)\) of portfolio on stocks and rate of consumption, is admissible for \((L, \varepsilon)\), where \(L\) is a subsistence random field, and write \((\pi, c) \in \mathcal{A}(L, \varepsilon)\) if for any \(x, s\) and \(p\) with \(x \geq L(s, s, p)\)

$$X(s, t, x, p) \geq L(s, t, p) \hspace{1cm} \text{for all } t.$$  \hspace{1cm} (9)

If there is not portfolio on stocks and rate of consumption for \((L, \varepsilon)\) we should say that the class cited above is empty, and we would denote this by \(\mathcal{A}(L, \varepsilon) = \emptyset\).

For any hedgeable wealth and income evolution structure \((X, E - C)\) with \((\pi, c)\) admissible for \((L, \varepsilon)\) it must hold that

$$x \geq \mathbb{E} \left[ H(s, T, p)L(s, T, p) + \int_s^T H(s, u, p) (c(s, u, x, p) - \varepsilon(s, u, p)) \, du \right]$$
for any $x \geq L(s, s, p)$, where the latter follows since the process defined by equation (10) is a super-martingale. It is often the case that $L(s, T, p) = 0$ for all $s$ and $p$. In these latter case the condition for the previous equation becomes

$$x \geq E \left[ \int_s^T H(s, u, p) (c(s, u, x, p) - \varepsilon(s, u, p)) du \right].$$

Next we explain the problem that we are interested to solve in this paper. We assume a subsistence random field $L$ and an endowment rate evolution structure $\varepsilon$. The control stochastic problem that we propose to solve concerns a small investor that at time $0$ has an initial capital $x$, is constrained to not let his wealth to fall below a subsistence random field $L(0, 0, p)$, has a rate of endowment process, $\varepsilon(0, 0, p)$ and has at his disposal portfolio/consumption processes $(\pi, c) \in A(L, \varepsilon)$. The following Proposition 1 is a direct consequence of Londoño [35], Theorem 2; it provides conditions under which $A(L, \varepsilon) \neq \emptyset$.

**Proposition 1.** Assume a subsistence random field $L$, and a rate evolution structure $\varepsilon$ as in Definition 4. Assume that $H(s, t, p) L(s, t, p) - \int_s^t H(s, u, p) \varepsilon(s, u, p) du$ is a martingale for all $s, p$. Then, there exists a rate of consumption and endowment evolution structure $(X, 0, \varepsilon)$ with $(\pi, 0) \in A(L, \varepsilon)$ where $\pi$ is the portfolio on stocks defined by $X$.

**Proof** Define $X$ by

$$X(s, t, x, p) \triangleq L(s, t, p) + (x - L(s, s, p)) H^{-1}(s, t, p)$$

It follows using Londoño [35], Theorem 2 that $(X, 0, \varepsilon)$ is the desired rate of consumption and endowment evolution structure. □

For the following condition let $(X, c, \varepsilon)$ be a rate of consumption and endowment evolution structure with discounted payoff process defined as

$$Y(s, t, x, p) \triangleq H(s, t, p) X(s, t, x, p) + \int_s^t H(s, u, p) (c(s, u, x, p) - \varepsilon(s, u, p)) du.$$  

(10)

**Condition 2.** Let $(X, c, \varepsilon)$ be a rate of consumption and endowment evolution structure, as above. We assume that there exist positive constants $\gamma \geq 1, \alpha_1, \alpha_2, \alpha_3, \beta_0, \cdots, \beta_n$, with $\alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1} + \sum_{i=0}^n \beta_i < 1$ such that the discounted payoff process $Y(s, t, x, p)$ satisfies

$$E \left[ | Y(x, p, s, t) - Y(x', p', s', t') | \gamma \right] \leq C \left( | s - s' |^{\alpha_1} + | t - t' |^{\alpha_2} + | x - x' |^{\alpha_3} + \sum_{i=0}^n | p_i - p_i' |^{\beta_i} \right).$$

This condition is usually satisfied when $X$ is a process that solves a stochastic differential equation. For instance, see Kunita [31], Lemma 4.5.6. The last inequality is needed in order to obtain a continuous modification of the random
field and its conditional expectation. See Kolmogorov’s continuity criterion for random fields (Kunita [31], Theorem 1.4.1 and Exercise 1.4.12). In this paper all rates of consumption and endowment evolution structures are assumed to satisfy condition 2, and conditional expectations of stochastic processes are the continuous modifications of the given stochastic processes.

For the problems of optimal consumption and terminal wealth that we describe below we shall assume that the subsistence random field is defined in a way such that the discounted (by the state price density process) subsistence random field of an agent can not fall below the current value of future endowments,

\[ L(s, t, p) = \frac{-1}{H(s, t, p)} E \left[ \int_t^T H(s, u, p) \epsilon(s, u, p) \, du \mid \mathcal{F}_{s,t} \right], \quad (11) \]

for \( 0 \leq s \leq t \leq T, \ p \in \mathbb{R}_n^+ \). In fact is not difficult to see that the family of stochastic processes defined by the last equation, is a subsistence random field with \( \mathcal{A}(L, \epsilon) \neq \emptyset \), since the discounted payoff process \( Y(s, t, p) \) satisfies

\[
Y(s, t, p) = H(s, t, p) L(s, t, p) - \int_s^t H(s, u, p) \epsilon(s, u, p) \, du
\]

and therefore is clearly a martingale.

In fact, Proposition 1 is a consequence of a more general theorem stated below. It allows to solve the problem of optimal consumption and investment under more general subsistence random fields.

**Theorem 1.** Let \((X, c, \epsilon)\) be a rate of consumption and endowment evolution structure as in Definition 3 with cumulative endowment and consumption structures \(C\), and \(E\) as defined by equations (7) and (8) respectively (satisfying Condition 2). If the family of processes defined by equation (10) are martingales for each \(x, p, s\) then \((X, E - C)\) is a hedgeable wealth-income structure.

**Proof** This is a straightforward consequence of Theorem 2 and Theorem 3 of Londoño [35]. \(\square\)

## 4 Consumption and Portfolio Optimization

In this paper we are interested in solving the optimization problems presented below. We assume a state preference structure \((U_1, U_2)\), and a subsistence random field \(L\), defined as in equation (11), with an endowment rate evolution structure \(\epsilon\).

**Problem 1** (Utility from consumption). Under the hypotheses assumed here the problem of maximizing expected utility from discounted consumption (by the state price density) is defined to be the problem of maximizing,

\[
V_1(x, p) \triangleq \sup_{(\pi, c) \in \mathcal{A}_1(L, x, x)} \mathbb{E} \int_0^T U_1(\pi(t), H(0, t, p)c(0, t, x, p)) \, dt,
\]
for all \( p \) and \( x > -E \left[ \int_0^T H(0, u, p)\varepsilon(0, u, p) \, du \right] \), where

\[
\mathcal{A}_1(L, \varepsilon, x) \triangleq \left\{ (\pi, c) \in \mathcal{A}(L, \varepsilon) : E \int_0^T U_1^-(t, H(0, t, p)c(0, t, x, p)) \, dt < \infty \right\}
\]

and \( U_1^-(t, x) = -(U_1(t, x) \wedge 0) \). We shall say that \( V_1 \) is the value function for this problem.

**Problem 2** (Utility from terminal wealth). Under the hypotheses assumed in this section the problem of maximizing expected utility from discounted terminal wealth is defined to be the problem of maximizing

\[
V_2(x, p) \triangleq \sup_{(\pi, c) \in \mathcal{A}_2(L, \varepsilon, x)} E \left[ U_2(H(0, T, p)X(0, T, x, p)) \right],
\]

for all \( p \) and \( x > -E \left[ \int_0^T H(0, u, p)\varepsilon(0, u, p) \, du \right] \), where

\[
\mathcal{A}_2(L, \varepsilon, x) \triangleq \left\{ (\pi, c) \in \mathcal{A}(L, \varepsilon) : E \left[ U_2^-(H(0, T, p)X(0, T, x, p)) \right] < \infty \right\}
\]

and \( U_2^-(x) = -(U_2(x) \wedge 0) \). We shall say that \( V_2 \) is the value function for this problem.

**Problem 3** (Utility from both consumption and terminal wealth). Under the hypotheses assumed above the problem of maximizing expected utility from both discounted consumption and discounted terminal wealth is defined to be the problem of maximizing

\[
V(x, p) \triangleq \sup_{(\pi, c, x) \in \mathcal{A}(L, \varepsilon, x)} E \left[ \int_0^T U_1^-(t, H(0, t, p)c(0, t, x, p)) \, dt + U_2^-(H(0, T, p)X(0, T, x, p)) \right]
\]

for all \( p \) and \( x > -E \left[ \int_0^T H(0, u, p)\varepsilon(0, u, p) \, du \right] \), where

\[
\mathcal{A}(L, \varepsilon, x) \triangleq \mathcal{A}_1(L, \varepsilon, x) \cap \mathcal{A}_2(L, \varepsilon, x).
\]

We say that \( V \) is the value function for this problem.

A few words are needed here. Using the valuation theory in Londoño [35] if \( X \in \mathcal{W}(\mathcal{M}) \) is a wealth evolution structure with the property that \( H(s, t, x, p)X(s, t, x, p) \) is an uniformly bounded below process (where the bound might depend on \( x, p, s \)) and such that

\[
x = E[H(s, T, p)X(s, T, x, p)] < \infty \quad (12)
\]

then \( E[H(0, T, x, p)X(0, T, x, p)] \) is the value at 0 of \( X \), in the sense that it is possible to find a portfolio that replicates \( X \) (see the definition of hedgeable wealth income structure). Let us notice that since \( H(s, T, p)X(s, T, x, p) \) is a supermartingale for all \( s, x, p \), under the assumption of integrability and boundness (from below) of this process, \( x \geq E[H(s, T, p)X(s, T, x, p)] \). Therefore in order to consider the optimization Problem 2 above it is sufficient to look at wealth
processes that satisfy equation (12). It follows that $H$ is the natural discount process that brings the current value of the wealth $X$ to present. As a consequence, if the technical conditions of Problem 2 are satisfied, an agent that solve this problem is using the utility as a way to value at time 0, his degree of happiness of holding (a final wealth) in the future. Notice that 0 is the time when the investment and consumption decisions are being made. Similar remarks hold for the other optimization problems.

We also point out that 0 does not play any special role, and the concepts like wealth, cumulative income, portfolio process, state preference structure, value functions and alike can be carried out for any time interval $[s, T]$ with $0 \leq s \leq T$. The above remark allows us to consider parameterized utility preference structures with parameter $0 \leq s \leq T$, defined on the time interval $[s, T]$. This models how an agent can change preferences as time evolves.

The problems consider above are different from the standard problems of optimal consumption and investment, see for instance (Karatzas and Shreve [27]). First, the optimization problems are over portfolio and consumptions which are consistent. Second, it looks at utility functions as reflecting the level of satisfaction over levels of consumption in Problem 1, final wealths in Problem 2, and on both in Problem 3, as valued by the market when the agent is making his consumption and investment decisions (at time 0).

Let us define

$$\Pi(s, t, p) \triangleq -E\left[ \int_t^T H(s, u, p) \varepsilon(s, u, p) \, du \mid \mathcal{F}_{s, t} \right]$$

(13)

For any $x > \Pi(t, t, p)$ we define $\mathcal{Y}(t, x, p)$ as the unique solution of

$$\mathcal{X}(t, \mathcal{Y}(t, x, p)) = x - \Pi(t, t, p)$$

where $\mathcal{X}$ is defined by equation (6). It follows that $\mathcal{Y}(t, x, p) = \mathcal{X}^{-1}(t, x - \Pi(t, t, p))$. Also, for the following theorem we introduce a function $G$ defined as

$$G(s, y) = \int_s^T U_1(t, I_1(t, y)) \, dt + U_2(I_2(y))$$

(14)

for $0 < y < \infty$ and $0 \leq s \leq T$.

**Theorem 2.** Assume the hypotheses of Problem 3, and in addition assume that $(U_1, U_2)$ is a homogeneous state preference structure (see Condition 1). Define $\xi$ as

$$\xi(s, t, x, p) \triangleq \begin{cases} H^{-1}(s, t, p) (\Pi(s, t, p) + \mathcal{X}(t, \mathcal{Y}(s, x, p))) & \text{if } x > \Pi(s, s, p) \\ H^{-1}(s, t, p) (\Pi(s, t, p) + x - \Pi(s, s, p)) & \text{otherwise}, \end{cases}$$

and let $c$ be defined as

$$c(s, t, x, p) \triangleq \begin{cases} H^{-1}(s, t, p) I_1(t, \mathcal{Y}(s, x, p)) & \text{if } x > \Pi(s, s, p) \\ 0 & \text{otherwise.} \end{cases}$$

Then, $(\xi, c, \varepsilon)$ is a hedgeable cumulative consumption and endowment structure, with portfolio $(\pi, c) \in \mathcal{A}(L, \varepsilon)$ that is optimal for the problem of optimal consumption and
investment. The corresponding optimal portfolio on stocks is

\[
[\xi(s, t, x, p) - \Pi(t, P(s, t, p)) - \phi_0(t, P(s, t, p))] (\sigma \sigma')^{-1}(b + \delta - r_1)(s, t, p) \\
- (\phi_1(t, P(s, t, p)), \cdots, \phi_n(t, P(s, t, p)))' \equiv \phi_0(t, P(s, t, p)) \equiv 0 \leq i \leq n.
\]  

(15)

where

\[
\Pi(t, p) \equiv \Pi(t, t, p) \\
and, \; \phi_i(t, p) \equiv p_i \frac{\partial \Pi(t, t, p)}{\partial p_i}
\]

Proof Let us point out that Condition 1 implies that \(\xi\) is a (consistent) process, and clearly it is Lipschitz continuous. The homogeneity also implies that \(c\) is a \((\xi, P)\) consistent process of class \(C^{0,1}\). We observe that

\[
Y(s, t, x, p) \equiv H(s, t, p)\xi(s, t, x, p) + \int_s^t H(s, u, p)(c(s, u, x, p) - \varepsilon(s, u, p)) \, du \\
= x + E\int_s^T H(s, u, p)\varepsilon(s, u, p) \, du - E\left[\int_s^T H(s, u, p)\varepsilon(s, u, p) \, du \mid F_{s,t}\right]
\]

is a martingale, and therefore Theorem 1 implies that \((\xi, c, \varepsilon)\) is a rate of consumption and endowment structure with portfolio \((\pi, c) \in A(L, \varepsilon)\). Next, we observe that for \(x > \Pi(s, t, p)\)

\[
E[\int_s^T U_1(t, H(s, t, p)c(s, t, x, p)) \, dt] + E[U_2(H(s, T, p)c(s, T, x, p))] = G(s, Y(s, x, p))
\]

where \(G\) is the function defined by equation (14). If \((X', \varepsilon, c')\) is a hedgeable rate of consumption, endowment and wealth evolution structure, then for \(x > \Pi(s, t, p)\),

\[
E[\int_s^T U_1(t, H(s, t, p)c'(s, t, x, p)) \, dt + U_2(H(s, T, p)c'(s, T, x, p))] \leq \\
G(s, Y(s, x, p)) - Y(s, x, p) \left[\int_s^T I_1(t, Y(s, x, p)) \, dt + I_2(Y(s, x, p))\right] \\
+ Y(s, x, p)E \left[H(s, T, p)c'(s, T, x, p) + \int_s^T H(s, u, p)(c'(s, u, x, p)) \, du\right] \\
\leq G(s, Y(s, x, p))
\]

where the first inequality is a consequence of equation (5) and the last inequality is a consequence of the fact that the process defined by equation (10) is a super-martingale for any hedgeable wealth-income structure.

Next we prove that the optimal portfolio satisfies equation (15). It is known that the corresponding optimal portfolio should satisfy

\[
\sigma'(s, t, p)\pi(s, t, x, p) = H^{-1}(s, t, p)\varphi(s, t, x, p) + \xi(s, t, x, p)\theta(s, t, p)
\]

where \(\varphi(s, t, x, p)\) is the process such that

\[
Y(s, t, x, p) = x + \int_s^t \varphi'(s, u, x, p) \, dW_s(u)
\]
and $Y$ is the discounted (by the state price density) payoff process defined by equation (10). Using the uniqueness of the decomposition of a continuous semi-martingale as local martingale and a process of bounded variation, Itô’s rule, and the fact that $\varepsilon$ is a $P$ consistent process, it follows by a straightforward computation that the optimal portfolio is given by equation (15).

□

Remark 1. If $\varepsilon$ is a $P$ consistent process where $E\left[\int_s^T H(s, u, p)\varepsilon(s, u, p) \, du \mid \mathcal{F}_s, s \right]$ is a deterministic function then the proof of the above theorem shows that the optimal portfolio is

$$\pi(s, t, x, p) = (\sigma\sigma')^{-1}(b + \delta - r\mathbf{1}_n)(s, t, p)\xi(s, t, x, p)$$

One important example of the above case is when there are not additional income to invest in the portfolio.

Remark 2. One of the consequences of Theorem 2, is that the solution to Problem 3 above, under the hypothesis that the state preference structure is homogeneous, is also homogeneous in the sense that we explain next. For any time $0 \leq s \leq T$ the solution $(\xi, c)$ of Problem 3 (as well as its associated optimal portfolio) satisfies the property that its restriction to the time interval $[s, T]$ is also optimal for the problem of optimal consumption and investment after time $s$ (where the definition of the solution to the problem has been outlined after the definition of the solution to the problems of optimal consumption and investment). The latter remark is a consequence of the proof of Theorem 2.

Remark 3. If $\xi(s, t, x, p) = H^{-1}(s, t, p) (\Pi(s, t, p) + x - \Pi(s, s, p))$, then $(\xi, 0)$ is admissible for $(L, \varepsilon)$, and it can be seen as an admissible wealth process obtained from initially having $x$ dollars and the income process derived from the salary $\varepsilon$. It follows that $\xi(s, t, x, p) = \xi(s, t, \Pi(s, s, p) \land x, p) + \alpha \varepsilon H^{-1}(s, t, p)(x - \Pi(s, s, p))^+$, where $x^+ = 0 \lor x$ is the positive part of a number. It also follows that $c(s, t, x, p) = H^{-1}(s, t, p)(\alpha \varepsilon / \alpha \varepsilon)(x - \Pi(s, s, p))^+ = \alpha \varepsilon / \alpha \varepsilon (\xi(x, t, x, p) - \xi(s, t, \Pi(s, s, p) \land x, p))$. These conclusions can be interpreted as follows. If an agent has an homogeneous state preference structure, and the agent acts as he is maximizing the expected utility from discounted consumption and discounted terminal wealth as explained in Problem 3, he creates two accounts. In the first account (that we call the regulatory account) he hedges the risk of maintaining the total wealth above the minimal required by society. He put all the money derived from his salary into this account. In the other account (the consuming account) he invest in a way that its current value (in the future) is a proportion of the initial value. He also consume for a period of time a proportion of the wealth in his consuming account, and this factor (of proportion) is $\alpha / \alpha^2$.

Next, we present without proof the solution to the problem of maximizing expected utility from discounted consumption. The proof is similar to the proof of Theorem 2, and is left to the reader. In order to state the following theorem we introduce

$$X_1(t, y) = \int_t^T I_1(t', y) \, dt'$$

and

$$G_1(t, y) = \int_t^T U_1(t', I_1(t', y)) \, dt'$$
for \( y > 0 \) and \( 0 \leq t \leq T \). We also set \( \mathcal{Y}_1(t,x,p) = X_1^{t-1}(t,x - \Pi(t,t,p)) \) for \( x > 0 \) and \( 0 \leq t \leq T \).

**Theorem 3.** Assume a homogeneous state preference structure \((U_1, U_2)\), and subsistence random field \( L \) defined as in equation (11), with an endowment rate evolution structure \( \varepsilon \). Define \( \xi_1 \) as

\[
\xi_1(s,t,x,p) = \begin{cases} 
H^{-1}(s,t,p) \left( \Pi(s,t,p) + \mathcal{X}(t, \mathcal{Y}_1(s,x,p)) \right) & \text{if } x > \Pi(s,s,p) \\
H^{-1}(s,t,p) \left( \Pi(s,t,p) + x - \Pi(s,s,p) \right) & \text{otherwise,}
\end{cases}
\]

and let \( c_1 \) be defined as

\[
c_1(s,t,x,p) = \begin{cases} 
H^{-1}(s,t,p) \mathcal{I}(t, \mathcal{Y}_1(s,x,p)) & \text{if } x > \Pi(s,s,p) \\
0 & \text{otherwise.}
\end{cases}
\]

Then, \((\xi_1, c_1, \varepsilon)\) is a rate of consumption and endowment structure, with portfolio \((\pi_1, c_1) \in \mathcal{A}(L, \varepsilon)\) that is optimal for problem 1. The optimal portfolio on stocks is

\[
\left[ \xi_1(s,t,x,p) - \Pi(t, P(s,t,p)) - \phi_0(t, P(s,t,p)) \right](\sigma \sigma')^{-1}(b + \delta - r)\mathbf{1}_n(s,t,p)
- (\phi_1(t, P(s,t,p)), \cdots, \phi_n(t, P(s,t,p)))' \]

where \( \Pi \), and \( \phi_i \), \( 0 \leq i \leq n \) are defined as in Theorem 2.

Also the solution for problem (11) is stated without a proof.

**Theorem 4.** Assume a homogeneous state preference structure and subsistence random field, \( L \) defined as in equation (11), with an endowment rate evolution structure \( \varepsilon \). Let \( \xi_2 \) be defined as

\[
\xi_2(s,t,x,p) = H^{-1}(s,t,p) \left( \Pi(s,t,p) + x - \Pi(s,s,p) \right)
\]

Then, \((\xi_2, 0, \varepsilon)\) is a rate of consumption and endowment structure, with portfolio \((\pi_2, 0) \in \mathcal{A}(L, \varepsilon)\) that is optimal for problem 2. The optimal portfolio on stocks is

\[
\left[ \xi_2(s,t,x,p) - \Pi(t, P(s,t,p)) - \phi_0(t, P(s,t,p)) \right](\sigma \sigma')^{-1}(b + \delta - r)\mathbf{1}_n(s,t,p)
- (\phi_1(t, P(s,t,p)), \cdots, \phi_n(t, P(s,t,p)))' \quad (17)
\]

where \( \Pi \), and \( \phi_i \), \( 0 \leq i \leq n \) are defined as in Theorem 2.

**Remark 4.** We point out that the wealth process and the portfolio process obtained in the solution of the problem 2, is independent of the utility function.

**References**


