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Reduced dynamics in the independent oscillator model: exact versus Born–Markov approximation

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Abstract

We derive the formal, exact reduced dynamics for the independent oscillator model in the rotating wave approximation at zero and finite temperature. We show that the traditional form of the Born–Markov approximation is valid beyond this limit, the effect of higher-order contributions being encapsulated into three time-dependent coefficients directly related to the time-dependent mean photon number, and two orthogonal field quadratures. A suggestion on how these coefficients can be measured in currently used optical cavities is given.

1. Introduction

The recent measurement of the decoherence process after the formation of a superposition of quantum states is a most remarkable achievement [1]. The theoretical work suggesting this experiment [2] models the field dissipation mechanism by a master equation. The approximations involved in the derivation of this equation are: (a) typical time scales of the environment dynamics are much shorter than the ones involving the field; (b) the field–environment coupling is weak. It is also well known that frequently master equations remain good approximations even when one is far from its validity limits. In the present contribution, we investigate why this is the case and show that if the system can be modelled by linearly coupled oscillators, the formal exact equation for the field's reduced density has precisely the same form as its

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Born–Markov limit, the difference being that the coefficients (effective frequency, dissipation and diffusion) are now time-dependent. This formal exact equation can be analytically solved and all field–environment coupling effects to all orders are encapsulated into three time-dependent coefficients, an instantaneous frequency shift, the instantaneous energy rate at zero temperature and the time-dependent average excitation number. Moreover, we show that these three ingredients are experimentally accessible with currently used setups in optical cavities. It is thus possible to experimentally evaluate the adequacy of the Born–Markov approximation, once the time-dependence of these coefficients is established.

In Section 2, we present the formal derivation of the exact equation governing the reduced field dynamics. In Section 3, we present analytical solutions for the time evolution of several initial conditions. The zero temperature limit is studied in Section 4. In Section 5 it is shown how to (implicitly) calculate the coefficients of the exact reduced equation in terms of the microscopic parameters. Section 6 contains a proposal for the experimental determination of these coefficients and the last section contains some conclusions and final remarks.

2. Formal derivation of the Field's reduced dynamics

Recently, the exact master equation for quantum Brownian motion in a general environment has been derived using both path integral techniques [3] and the tracing of the evolution equation for the Wigner function [4,5]. We closely follow the latter approach to derive the exact master equation for the oscillator-independent model in the RWA at zero temperature with a factorized initial condition. The Hamiltonian of the model is

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}) + \hbar\sum_k \omega_k(a_k^\dagger a_k + \tfrac{1}{2}) + \hbar\sum_k c_k(a^\dagger a_k + a_k^\dagger a) \ . \tag{1}$$

The solution for an initial value problem allows for an easy visualization in contrast to the model without the RWA approximation. Besides its intrinsic interest as an exactly soluble model, Hamiltonian (1) may be useful in treating leaking Bose–Einstein condensates [6], in materials with modified dispersion relations [7], or in any case of non-ohmic strength function, where the Born–Markov approximation is not adequate [6].

We assume that at $t=0$ the total density operator is given by

$$\rho(0) = \rho_S(0)\otimes\prod_k \frac{\mathrm{e}^{-\beta\hbar\omega_k a_k^\dagger a}}{\mathrm{Tr}\,\mathrm{e}^{-\beta\hbar\omega_k a_k^\dagger a}} \stackrel{\beta\to\infty}{\to} \rho_S(0)\otimes\prod_k |0_k\rangle\langle 0_k| \ , \tag{2}$$

where the subscript S, for system, refers to the main oscillator. The bath, i.e., the set of oscillators labelled by k, is initially in thermal equilibrium at inverse temperature β. At zero temperature, the tensor product of the vacuum of the main oscillator with the vacuum of the set of oscillators is the ground state of (1).

It is well known that for quadratic Hamiltonian, the Wigner function satisfies the classical Liouville equation. To obtain the classical Hamiltonian corresponding to (1), one uses the correspondence rule $a_\mu^{(\dagger)} \to \alpha_\mu^{(*)}$, where

$$\alpha_\mu^{(*)} = \sqrt{\frac{m_\mu \omega_\mu}{2\hbar}}\, q_\mu + (-)\, \frac{\mathrm{i}}{\sqrt{2\hbar m_\mu \omega_\mu}}\, p_\mu \tag{3}$$

and discards zero energy contributions. We use $\mu = 0, 1, 2, \ldots$, $k = 1, 2, \ldots$ and $a_0 \equiv a, \omega_0 = \omega$. From here on greek subindices denote non-negative integers while latin subindices denote positive integers. Using these conventions, the classical Liouville equation reads

$$\frac{\partial W(\alpha_\mu, \alpha_\mu^*, t)}{\partial t} = \frac{1}{\mathrm{i}\hbar} \sum_\mu \frac{\partial H}{\partial \alpha_\mu} \frac{\partial W}{\partial \alpha_\mu^*} - \frac{1}{\mathrm{i}\hbar} \sum_\mu \frac{\partial H}{\partial \alpha_\mu^*} \frac{\partial W}{\partial \alpha_\mu} , \tag{4}$$

where $W(\alpha_\mu, \alpha_\mu^*, t)$ is the Wigner function in the quantum case and the probability density function in the classical case. More explicitly,

$$\begin{aligned} \frac{\partial W(\alpha_\mu, \alpha_\mu^*, t)}{\partial t} &= -\mathrm{i}\omega\alpha^* \frac{\partial W}{\partial \alpha^*} + \mathrm{i}\omega\alpha \frac{\partial W}{\partial \alpha} - \mathrm{i} \sum_k \omega_k \alpha_k^* \frac{\partial W}{\partial \alpha_k^*} \\ &\quad + \mathrm{i} \sum_k \omega_k \alpha_k \frac{\partial W}{\partial \alpha_k} - \mathrm{i} \sum_k c_k \alpha_k^* \frac{\partial W}{\partial \alpha^*} + \mathrm{i} \sum_k c_k \alpha_k \frac{\partial W}{\partial \alpha} \\ &\quad - \mathrm{i} \sum_k c_k \alpha^* \frac{\partial W}{\partial \alpha_k^*} + \mathrm{i} \sum_k c_k \alpha \frac{\partial W}{\partial \alpha_k} . \end{aligned} \tag{5}$$

The Wigner function corresponding to the initial condition, Eq. (2) is given by

$$\begin{aligned} W(\alpha_\mu, \alpha_\mu^*, t = 0) = W^0(\alpha_\mu, \alpha_\mu^*) &= W_S^0(\alpha, \alpha^*) W_B^0(\alpha_k, \alpha_k^*) \\ &= W_S^0(\alpha, \alpha^*) \prod_k N_k \mathrm{e}^{-2\tanh(\hbar\omega_k \beta)\alpha_k \alpha_k^*} , \end{aligned} \tag{6}$$

where the N_k's are normalization constants. Integrating Eq. (5) over the bath variables yields

$$\frac{\partial \tilde{W}(\alpha, \alpha^*, t)}{\partial t} = -\mathrm{i}\omega\alpha^* \frac{\partial \tilde{W}}{\partial \alpha^*} + \mathrm{i}\omega\alpha \frac{\partial \tilde{W}}{\partial \alpha} - \mathrm{i} \frac{\partial G^*}{\partial \alpha^*} + \mathrm{i} \frac{\partial G}{\partial \alpha} \tag{7}$$

with

$$\tilde{W}(\alpha, \alpha^*, t) = \int \left(\prod_k \mathrm{d}\alpha_k \, \mathrm{d}\alpha_k^* \right) W(\alpha_\mu, \alpha_\mu^*, t) , \tag{8}$$

and

$$G(\alpha, \alpha^*, t) = \int \left(\prod_k \mathrm{d}\alpha_k \, \mathrm{d}\alpha_k^* \right) \sum_k c_k \alpha_k W(\alpha_\mu, \alpha_\mu^*) . \tag{9}$$

As done in Ref. [5], it is easy to show that $G(\alpha, \alpha^*, t)$ can be written in terms of $\tilde{W}$. We remember that for quadratic Hamiltonians $W(\alpha_\mu, \alpha_\mu^*, t) = W^0(\alpha_\mu(-t), \alpha_\mu^*(-t))$, where $\alpha_\mu(t)$ is the solution of the classical equations of motion. If we define $\vec{\alpha}(t) = (\alpha_0(t), \alpha_1(t), \ldots)$ and denote its transpose by $\vec{\alpha}^{\mathrm{T}}(t)$, we have

$$\vec{\alpha}^{\mathrm{T}}(t) = V(t)\vec{\alpha}^{\mathrm{T}}(0), \qquad \vec{\alpha}^*(t) = (\vec{\alpha}(t))^* \,, \tag{10}$$

where V is a unitary matrix. As usual, V can be diagonalized by some matrix U, such that $V(t) = U^\dagger \Delta(t) U$ with Δ diagonal, ($\Delta = \mathrm{diag}(\ldots, \mathrm{e}^{-\mathrm{i}\omega_\mu t}, \ldots)$). Taking the Fourier transform of G, and changing variables from $\{\alpha_\mu(-t), \alpha_\mu^*(-t)\}$ to $\{\alpha_\mu(0), \alpha_\mu^*(0)\}$, with unit Jacobian, we obtain

$$\begin{aligned} G(\kappa,\kappa') = \int \prod_\mu \mathrm{d}\alpha_\mu(0)\,\mathrm{d}\alpha_\mu^*(0) \mathrm{e}^{\mathrm{i}\kappa \sum_\nu p_\nu(t)\alpha_\nu(0)} \mathrm{e}^{\mathrm{i}\kappa' \sum_\nu p_\nu^*(t)\alpha_\nu^*(0)} \\ \times \sum_\nu q_\nu(t)\alpha_\nu(0) W_S^0(\alpha^{(*)}(0)) W_B^0(\alpha_k^{(*)}(0)) \,, \end{aligned} \tag{11}$$

with $\{p_\nu, p_\nu^*, q_\nu\}$ time-dependent parameters. From Eq. (11), it is easy to see that the multiplication by $\alpha_0 = \alpha$ is equivalent to a derivation with respect to k, plus terms corresponding to multiplication by α_k, up to time-dependent coefficients. These last terms, as can be seen from (6), correspond to derivations with respect to (w.r.t.) α_k^*. This operation is equivalent to multiplication by k', as shows a simple integration by parts. Taking the inverse Fourier transform, we obtain a multiplication by α and a derivation w.r.t. α^*. Thus, noting that the Fourier transform of $\tilde{W}(\alpha,\alpha^*)$, $\tilde{W}(\kappa,\kappa')$ is given by

$$\begin{aligned} \tilde{W}(\kappa,\kappa') = \int \prod_\mu \mathrm{d}\alpha_\mu(0)\,\mathrm{d}\alpha_\mu^*(0) \mathrm{e}^{\mathrm{i}\kappa \sum_\nu p_\nu(t)\alpha_\nu(0)} \mathrm{e}^{\mathrm{i}\kappa' \sum_\nu p_\nu^*(t)\alpha_\nu^*(0)} \\ \times W_S^0(\alpha^{(*)}(0)) W_B^0(\alpha_k^{(*)}(0)) \,, \end{aligned}$$

we obtain

$$\begin{aligned} \mathrm{i}\,\frac{\partial G}{\partial \alpha} - \mathrm{i}\,\frac{\partial G^*}{\partial \alpha^*} = \mathrm{i}Y\,\frac{\partial}{\partial \alpha}\left(\alpha\tilde{W}\right) - \mathrm{i}Y^*\,\frac{\partial}{\partial \alpha^*}\left(\alpha^*\tilde{W}\right) \\ +(\mathrm{i}Z - \mathrm{i}Z^*)\,\frac{\partial^2 \tilde{W}}{\partial\alpha\,\partial\alpha^*} \,, \end{aligned} \tag{12}$$

with time-dependent functions Y, Z. Therefore, the Wigner equation can be written as

$$\begin{aligned} \frac{\partial \tilde{W}(\alpha,\alpha^*,t)}{\partial t} = -\mathrm{i}(\omega+\delta)\left(\alpha^*\,\frac{\partial\tilde{W}}{\partial\alpha^*} - \alpha\,\frac{\partial\tilde{W}}{\partial\alpha}\right) + 2\lambda\tilde{W} \\ +\lambda\left(\alpha^*\,\frac{\partial\tilde{W}}{\partial\alpha^*} + \alpha\,\frac{\partial\tilde{W}}{\partial\alpha}\right) + \lambda'\,\frac{\partial^2\tilde{W}}{\partial\alpha\partial\alpha^*} \,, \end{aligned} \tag{13}$$

where we have set $\mathrm{i}Y = \lambda + \mathrm{i}\delta$ and $\mathrm{i}Z - \mathrm{i}Z^* = \lambda'$. All of λ, δ and λ' are real functions. By comparing the system of equations found from both (5) and (13) we get

$$(\lambda + \mathrm{i}\delta)\langle\alpha\rangle = \mathrm{i}\sum_k c_k\langle\alpha_k\rangle \,, \tag{14}$$

$$(\lambda + \mathrm{i}\delta)\langle \alpha^2 \rangle = \mathrm{i} \sum_k c_k \langle \alpha\alpha_k \rangle \ , \tag{15}$$

$$-2\lambda' \langle \alpha\alpha^* \rangle = \mathrm{i} \sum_k c_k \langle \alpha\alpha_k^* \rangle - \mathrm{i} \sum_k c_k \langle \alpha^* \alpha_k \rangle \ . \tag{16}$$

We know that the solution of the Heisenberg equations can be written as follows:

$$a(t) = \eta(t) a(0) + \sum_\kappa \gamma_k(t) a_k(0) \ , \tag{17}$$

$$a_k(t) = \eta_k(t) a(t) + \sum_l \gamma_{kl} a_l(0) \ . \tag{18}$$

Using the above solution of the Heisenberg equations, and the fact that all of the first and second (symmetric) moments involving bath operators are zero, with the exception of $\langle \{a_k^\dagger, a_k\} \rangle = 2n_k + 1 = 2\coth(\hbar\omega_k \beta/2)$, we obtain

$$\lambda(t) + \mathrm{i}\delta(t) = \mathrm{i} \sum_k c_k \eta_k(t) \ , \tag{19}$$

$$\lambda'(t) = \sum_k c_k \left(\eta_{kl}(\beta) + \tfrac{1}{2} \right) (\mathrm{i}\gamma_l \gamma_{kl}^* - \mathrm{i}\gamma_l^* \gamma_{kl}) \ . \tag{20}$$

For reasons that will be clear soon, we write the diffusion coefficient $\lambda'(t)$ as $\lambda(t) + \varepsilon(t, \beta)$, with $\lim_{\beta \to \infty} \varepsilon(t, \beta) = 0$, as shown in Section 5. At zero temperature, since the tensor product of vacua is the ground state of (1), the corresponding (reduced) Wigner function $W_S(\alpha, \alpha^*)$ should be a stationary solution of the Wigner equation. When this condition is applied to Eq. (13), we obtain $\lambda = \lambda'$. If they were not equal it would imply the non-existence of an exact master equation. However, this is not the case, as we show next.

It is not hard to show that the operator equation for the system's reduced density operator, equivalent to the Wigner equation (13), is given by

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} = &\frac{1}{\mathrm{i}\hbar} [\hbar(\omega + \delta) a^\dagger a, \rho] + (\lambda + \varepsilon)(2a \bullet a^\dagger - a^\dagger a \bullet - \bullet\, a^\dagger a)\rho \\ &+ \varepsilon(2a^\dagger \bullet a - a a^\dagger \bullet - \bullet\, a a^\dagger)\rho = \mathscr{L}(t)\rho(t) \ , \end{aligned} \tag{21}$$

where the usual dot superoperator convention has been used. The usual Born–Markov RWA master equation is of this form with constant coefficients [8]. Some results can be obtained at once from (21): premultiplying by a and taking the trace we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle a \rangle = \frac{\mathrm{d}\alpha}{\mathrm{d}t} = (-\mathrm{i}(\omega + \delta) - \lambda)\alpha \ , \tag{22}$$

which can be immediately solved to give

$$\alpha(t) = \exp(-\mathrm{i}\Omega(t) - \Lambda(t))\alpha(0) \tag{23}$$

with

$$\Omega(t) = \int_0^t \mathrm{d}\tau(\omega + \delta)(\tau), \quad \Lambda(t) = \int_0^t \mathrm{d}\tau\lambda(\tau) \, . \tag{24}$$

Note that this result is *independent* of ε, i.e., it does not depend on the temperature. Premultiplying (21) by $a^\dagger a$, and taking the trace we get the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle a^\dagger a\rangle(t) = -2\lambda\langle a^\dagger a\rangle(t) + 2\varepsilon \tag{25}$$

whose solution is

$$\begin{aligned}\langle a^\dagger a\rangle(t) &= \exp(-2\Lambda(t))\langle a^\dagger a\rangle(0) + \mathcal{N}(t)\\ &= \langle a^\dagger a\rangle(t; \beta \to \infty) + 2\exp(-2\Lambda(t)) \int_0^t \mathrm{d}\tau\varepsilon(\tau)\exp(2\Lambda(\tau)) \, ,\end{aligned} \tag{26}$$

where it is evident that $\mathcal{N}(t) = 2\exp(-2\Lambda(t)) \int_0^t \mathrm{d}\tau\varepsilon(\tau)\exp(2\Lambda(\tau))$ vanishes in the zero temperature limit, when $\varepsilon(t)$ goes to zero. Contrary to the exact equations found in [3–5], where the coefficients of the exact reduced master equation require a more involved interpretation, the real functions that appear in the master equation admit easy physical explanation: $\delta(t)$ is the instantaneous frequency shift, $\lambda(t)$ is the instantaneous energy rate of change at zero temperature and $\varepsilon(t)$ is the instantaneous energy rate of change at finite temperature but with the system initially in the vacuum state. Moreover $\mathcal{N}(t)$, which is related to both $\varepsilon(t)$ and $\delta(t)$ is the mean number of excitations when the initial state of the system is the ground state.

3. The evolution superoperator and some initial states

We can use Lie algebraic methods [9] to find the evolution superoperator $\mathcal{U}$. Indeed, we can verify that the superoperators $\mathcal{M} = a^\dagger a\bullet$, $\mathcal{P} = \bullet a^\dagger a$, $\mathcal{J} = a \bullet a^\dagger$ and $\mathcal{R} = a^\dagger \bullet a$ form an algebra,

$$[\mathcal{M}, \mathcal{P}] = 0, \quad [\mathcal{M}, \mathcal{J}] = -\mathcal{J} = [\mathcal{P}, \mathcal{J}], \quad [\mathcal{M}, \mathcal{R}] = -\mathcal{R} = [\mathcal{P}, \mathcal{R}] \, . \tag{27}$$

Thus, in order to obtain an analytical solution we first write $\mathcal{U}(t) = v\mathrm{e}^{w\mathcal{R}}\mathrm{e}^{x\mathcal{M}}\mathrm{e}^{y\mathcal{P}}\mathrm{e}^{z\mathcal{J}}$. Next we derive this expression with respect to time using the formula $\exp(xA)B \times \exp(-xA) = B + x[A, B] + x^2[B, [B, A]]/2! + \cdots$ and commutation relations (27). Comparing coefficients in the equation $\mathrm{d}\mathcal{U}/\mathrm{d}t = \mathcal{L}(t)\mathcal{U}(t)$, and solving the resulting differential equations we obtain

$$\begin{aligned} v(t) &= \frac{1}{1 + \mathcal{N}(t)}, \quad \omega(t) = \frac{\mathcal{N}(t)}{1 + \mathcal{N}(t)},\\ x(t) &= -\mathrm{i}\Omega(t) - \Lambda(t) - \tfrac{1}{2}\ln(1 + \mathcal{N}(t)) = y^*(t),\\ z(t) &= 1 - \frac{\exp(-2\Lambda(t))}{1 + \mathcal{N}(t)} \, . \end{aligned} \tag{28}$$

Let us suppose that ρ satisfies the equation $\mathrm{d}\rho/\mathrm{d}t = \mathscr{L}(X_i\bullet, \bullet X_i; t)\rho(t)$, where the X_i are operators (note that $\mathscr{L}$ is a general linear superoperator), and that ρ can be written as $U\rho' U^{-1}$. Then, ρ' satisfies the equation $\mathrm{d}\rho'/\mathrm{d}t = \mathscr{L}'\rho'(t)$ If, moreover, we choose $U = \exp(\sigma a^\dagger - \sigma^* a) = D(\sigma)$, the displacement operator, and $\mathscr{L}$ is that of the RWA, then

$$\mathscr{L}'(t) = \mathscr{L}(t) + \left((\mathrm{i}\dot{\Omega} + \lambda)\sigma + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)(a^\dagger \bullet - \bullet\, a^\dagger) - \left((-\mathrm{i}\dot{\Omega} + \lambda)\sigma^* + \frac{\mathrm{d}\sigma^*}{\mathrm{d}t}\right)(a \bullet - \bullet\, a) \,. \tag{29}$$

It is easy to see that if

$$\sigma(t) = \sigma(0)\exp(-\mathrm{i}\Omega(t) - \Lambda(t)), \qquad \sigma^*(t) = (\sigma(t))^* \,, \tag{30}$$

then both ρ and ρ' satisfy the same equation. That is, we have shown that $D(\sigma(t))\rho(t) \times D^\dagger(\sigma(t))$ satisfies Eq. (21) whenever $\rho(t)$ does the same. We remark that this result does *not* depend on the temperature of the bath.

We now turn to the evaluation of the density matrix evolved with the superoperator found above for some initial states of relevance to quantum optics.

3.1. Time evolution of the ground state

Specifying now the initial state as the ground state $\rho(0) = |0\rangle\langle 0|$ we will need, as intermediate steps,

$$\mathrm{e}^{x\mathscr{M}}|0\rangle\langle 0| = \mathrm{e}^{y\mathscr{P}}|0\rangle\langle 0| = \mathrm{e}^{z\mathscr{I}}|0\rangle\langle 0| = |0\rangle\langle 0| \tag{31}$$

and

$$\mathrm{e}^{x\mathscr{R}}|0\rangle\langle 0| = \sum_0^\infty \frac{x^n}{n!}(a^\dagger)^n|0\rangle\langle 0|a^n = \sum_0^\infty x^n|n\rangle\langle n| \,. \tag{32}$$

Using the above results, we get

$$\rho(t) = \mathscr{U}(t)|0\rangle\langle 0| = \sum_0^\infty \frac{1}{1+\mathscr{N}(t)}\left(1 + \frac{1}{\mathscr{N}(t)}\right)^n |n\rangle\langle n| = \sum_0^\infty P_n(t)|n\rangle\langle n| \,. \tag{33}$$

The above formula displays the so-called decomposition in natural orbits [10] where the quantities P_n can be directly interpreted as probabilities. We can write the evolved density matrix in the alternative form $\rho(t) = \exp((1 + 1/\mathscr{N})a^\dagger a)/(1 + \mathscr{N})$, which is the form of an instantaneous thermal density matrix, with $\mathscr{N}(t) = \langle a^\dagger a\rangle(t)$. Had we chosen an initial thermal state, with mean number of excitations $\bar{n}(0)$, the density matrix would have remained a thermal state, but now $M(t) = \bar{n}(0)\exp(-2\Lambda(t)) + \mathscr{N}(t)$. If we use the instantaneous oscillator frequency $\omega' = \omega + \delta$, it is possible to define an instantaneous temperature through the relation $T(t) = \hbar(\omega + \delta)/(k_B \ln(1 + 1/\mathscr{N}))$, with k_B the Boltzmann constant. Moreover, we have obtained a physical interpretation for the quantity $\mathcal{N}(t)$: it is the mean number of excitations of the main oscillator at time t when it was initially prepared in its ground state.

3.2. Time evolution of an initial Fock state

To calculate the density matrix for an initial Fock state, it is better to write the evolution superoperator in the form $\mathcal{U}(t) = v\exp(w\mathcal{R})\exp(z'\mathcal{J})\exp(x\mathcal{M})\exp(y\mathcal{P})$, where w, x, y are given in (28), and $z'(t) = (1+\mathcal{N})\exp(2\Lambda)$. We use

$$\mathrm{e}^{z\mathcal{J}}\mathrm{e}^{x\mathcal{M}}\mathrm{e}^{y\mathcal{P}}|m\rangle\langle m| = \sum_{k=0}^{m}(\mathrm{e}^{x+y})^{m-k}(z\mathrm{e}^{x+y})^{k}\,\frac{m!}{(m-k)!k!}\,|m-k\rangle\langle m-k|$$

$$= \sum_{k=0}^{m}\frac{m!}{(m-k)!k!}\,(\mathrm{e}^{x+y})^{k}(z\mathrm{e}^{x+y})^{m-k}|k\rangle\langle k| \tag{34}$$

and

$$\mathrm{e}^{u\mathcal{R}}|m\rangle\langle m| = \sum_{k=0}^{\infty}\frac{m!u^k}{(m-k)!k!}\,|k\rangle\langle k| \tag{35}$$

to see that the density matrix at time t is given by $\rho(t) = \sum_{k=0}^{\infty} P_{m,s}(t)|s\rangle\langle s|$, with

$$P_{m,s}(t) = \frac{\mathrm{e}^{-2m\Lambda}}{(1+\mathcal{N})^{m+1}}\,\frac{m!}{s!}\sum_{k=0}^{\min(m,s)}\frac{([1+\mathcal{N}]\mathrm{e}^{2\Lambda}-1)^k}{(m-k)!(s-k)!}\left(1+\frac{1}{\mathcal{N}}\right)^{s-k}. \tag{36}$$

Since the former density matrix has been expressed in terms of natural orbits, the quantities $P_{m,s}(t)$ are readily interpreted as probabilities. The transformation property discussed above allows us to write the evolution of an initial generalized coherent state $|\sigma m\rangle = D(\sigma)|m\rangle$, where D is the displacement operator and $|n\rangle$ the nth number state. We have $\mathcal{U}(t)|\sigma_0 m\rangle\langle\sigma_0 m| = \sum_{k=0}^{\infty} P_{m,s}(t)|\sigma(t)s\rangle\langle\sigma(t)s|$, with $P_{m,s}(t)$ given by (36) and $\sigma(t)$ by (30).

One interesting point to be investigated is if there exists an asymptotic density operator. Provided that our environment is such that $\lim_{t\uparrow\infty}\Lambda(t)\to\infty$ and $\lim_{t\uparrow\infty}\mathcal{N}(t) = n_\infty$, the asymptotic evolution superoperator can be written as

$$\lim_{t\uparrow\infty}\mathcal{U}(t) = \frac{1}{1+n_\infty}\exp\left(\frac{n_\infty}{1+n_\infty}\mathcal{R}\right)(|0\rangle\langle 0|\bullet)\exp(\mathcal{J})(\bullet|0\rangle\langle 0|)\,, \tag{37}$$

which applied to a generic normalized initial density $\rho(0)$ gives

$$\rho_\infty = \frac{1}{1+n_\infty}\exp(\mathcal{J})\rho(0)\langle 0|\exp\left(\frac{n_\infty}{1+n_\infty}\mathcal{R}\right)|0\rangle\langle 0|$$

$$= \frac{1}{1+n_\infty}(\mathrm{Tr}\,\rho(0))\exp\left[\left(1+\frac{1}{n_\infty}\right)a^\dagger\bullet a\right]|0\rangle\langle 0|$$

$$= \frac{1}{1+n_\infty}\exp\left[\left(1+\frac{1}{n_\infty}\right)a^\dagger\bullet a\right]|0\rangle\langle 0| \tag{38}$$

$$= \frac{1}{1+n_\infty}\sum_{n=0}^{\infty}\exp\left[\left(1+\frac{1}{n_\infty}\right)n\right]|n\rangle\langle n| \; . \tag{39}$$

Thus, whenever the established conditions are met, the density operator approaches asymptotically to a thermal state with a mean number of excitations equal to that of the environment. The existence of a unique asymptotic density cannot be taken for granted: in the model of decoherence without damping studied in Ref. [11] even when the coefficient of decoherence grows indefinitely with time, the asymptotic state depends on the initial state.

The normal-order characteristic functional $C^{(n)}(\xi,\xi^\dagger,t)$ given by [8]

$$\begin{aligned} C^{(n)}(\xi,\xi^\dagger,t) &= \mathrm{Tr}\, \mathrm{e}^{\mathrm{i}\xi a^\dagger}\mathrm{e}^{\mathrm{i}\xi^* a}\rho(t) = \mathrm{Tr}\, \mathrm{e}^{\mathrm{i}\xi a^\dagger}\mathrm{e}^{\mathrm{i}\xi^* a}\mathscr{U}(t)\rho(0) \\ &= \mathrm{Tr}\, \mathscr{U}^\dagger(t)\mathrm{e}^{\mathrm{i}\xi a^\dagger}\mathrm{e}^{\mathrm{i}\xi^* a}\rho(0) \; , \end{aligned} \tag{40}$$

where $\mathscr{U}(t)$ is the evolution superoperator and $\mathscr{U}^\dagger(t)$ its adjoint, is the generating functional of the normally ordered moments. After a somewhat lengthy but straightforward calculation, we obtain

$$C^{(n)}(\xi,\xi^\dagger,t) = \mathrm{e}^{-2\mathscr{N}\xi x \mathrm{i}^\dagger}\,\mathrm{Tr}\,\mathrm{e}^{\mathrm{i}\xi\exp(-\Lambda+\mathrm{i}\Omega)a^\dagger}\mathrm{e}^{\mathrm{i}\xi^*\exp(-\Lambda-\mathrm{i}\Omega)a}\rho(0) \; . \tag{41}$$

If we calculate $\langle a\rangle(t)$ and $\langle a \dagger a\rangle(t)$ using

$$\langle a\rangle(t) = \frac{\partial}{\partial\xi^*}C^{(n)}(\xi,\xi^\dagger,t)|_{\xi=0=\xi^\dagger}, \quad \langle a \dagger a\rangle(t) = \frac{\partial^2}{\partial\xi\partial\xi^*}C^{(n)}(\xi,\xi^\dagger,t)|_{\xi=0=\xi^\dagger} \; , \tag{42}$$

we arrive at the same results as before. These can be compared to those obtained in Ref. [12].

4. The zero temperature limit

The zero temperature limit has its own special interest, both as an approximation at low temperatures, and as the relevant case for leaking Bose–Einstein condensates. In this case, the evolution superoperator can be expressed as $\mathscr{U}(t) = \mathrm{e}^{\tilde{x}\mathscr{M}}\mathrm{e}^{\tilde{y}\mathscr{P}}\mathrm{e}^{\tilde{x}\mathscr{J}}$ with

$$\tilde{x}(t) = -\mathrm{i}\Omega(t) - \Lambda(t) = \tilde{y}^{\,*}(t), \quad \tilde{z}(t) = 1 - \exp(-2\Lambda(t)) \; . \tag{43}$$

Since the vacuum is solution of the master equation at zero temperature, the transformation property discussed above (30) indicates that

$$\begin{aligned} D(\sigma(t))|0\rangle\langle 0|D^\dagger(\sigma(t)) &= |\sigma(t)\rangle\langle\sigma(t)| = |\sigma(0)\exp(-\mathrm{i}\Omega(t)-\Lambda(t))\rangle \\ &\quad\times\langle\sigma(0)\exp(-\mathrm{i}\Omega(t)-\Lambda(t))| = \mathscr{U}(t)|\sigma_0\rangle\langle\sigma_0| \end{aligned} \tag{44}$$

also solves the master equation. That means that, whenever the interaction system–environment is modelled adequately by Hamiltonian (1), initial coherent states evolve preserving their coherence, no matter what the details of the coupling strength with individual degrees of freedom of the environment. A measurement of the norm and the phase of an initial coherent state is enough to determine the functions Ω and Λ, and hence, in principle, to determine the evolution of any other initial state.

The evolution of an initial Fock state is also easily calculated (for previous work see Ref. [13]) as follows:

$$\begin{aligned}\mathscr{U}(t)|m\rangle\langle m| &= \sum_{k=0}^{m} p_{k,m}(t)|k\rangle\langle k| \\ &= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!}\,(\mathrm{e}^{-2\Lambda(t)})^k(1-\mathrm{e}^{-2\Lambda(t)})^{m-k}|k\rangle\langle k| \;. \end{aligned} \tag{45}$$

This solution can be immediately generalized: if we have an initial generalized coherent state, $|m;\sigma_0\rangle = D(\sigma_0)|m\rangle$, it evolves into a mixture of generalized coherent states, as given by

$$\mathscr{U}(t)|m;\sigma_0\rangle\langle m;\sigma_0| = \sum_{k=0}^{m} p_{k,m}(t)|k;\sigma(t)\rangle\langle k;\sigma(t)| \;. \tag{46}$$

Applying the evolution operator $\mathscr{U}(t)$ to an initial density matrix element $|\sigma_0\rangle\langle\sigma_0'|$, one obtains

$$\mathscr{U}(t)|\sigma_0\rangle\langle\sigma_0'| = \frac{\langle\sigma_0'|\sigma_0\rangle}{\langle\sigma'(t)|\sigma(t)\rangle}\,|\sigma(t)\rangle\langle\sigma'(t)| \;. \tag{47}$$

Note that a heuristic argumentation also leads to (47). Indeed, since a coherent state remains coherent, we expect $\mathscr{U}(t)|\sigma_0\rangle\langle\sigma_0'|=N(t)|\sigma(t)\rangle\langle\sigma'(t)|$. As long as the exact dynamical equation for ρ preserves the trace, $\mathrm{d}/\mathrm{d}t\,\mathrm{Tr}\rho_S(t)=\mathrm{Tr}(\mathrm{d}\rho_S(t)/\mathrm{d}t)=\mathrm{Tr}(\mathscr{L}(t)\rho_S(t))=0$, the normalization factor $N(t)$ cannot be other than that of Eq. (47). Schrödinger cat states, superposition of coherent states, have been extensively studied due to their quantum interference characteristics, and the unique possibility they offer to follow the quantum to classical dynamical transition. In particular, in Refs. [2,14] the evolution of an initial cat state in the presence of a Markovian bath was calculated. Here, we extend this result for non-Markovian baths, observing that the evolution of an initial even ($\rho_{\sigma_0 e}$) or odd ($\rho_{\sigma_0 o}$) superposition state can be calculated from (47) and (44). Indeed,

$$\rho_{\sigma_0 e(o)} = N_{e(o)}(\sigma_0)(|\sigma_0\rangle, |-\sigma_0\rangle)\begin{pmatrix} 1 & (-)1 \\ (-)1 & 1 \end{pmatrix}\begin{pmatrix} \langle\sigma_0| \\ \langle-\sigma_0| \end{pmatrix} \;, \tag{48}$$

where $N_{e(o)}(\sigma_0)=(1+(-)\langle-\sigma_0|\sigma_0\rangle)^{-1}/2$ is a normalization factor, evolves as follows:

$$\mathscr{U}\rho_{\sigma_0 e(o)} = N_{e(o)}(\sigma_0)(|\sigma_t\rangle, |-\sigma_t\rangle)\begin{pmatrix} 1 & \frac{(-)\langle-\sigma_0|\sigma_0\rangle}{\langle-\sigma(t)|\sigma(t)\rangle} \\ \frac{(-)\langle-\sigma_0|\sigma_0\rangle}{\langle-\sigma(t)|\sigma(t)\rangle} & 1 \end{pmatrix}\begin{pmatrix} |\sigma_t\rangle \\ |-\sigma_t\rangle \end{pmatrix} \;. \tag{49}$$

We can rewrite Eq. (49) in a more convenient way, in terms of natural orbitals, as

$$\mathscr{U}(t)\rho_{\sigma_0 e(o)} = p^{e(o)}_{e(o)}(t)\rho_{\sigma(t)e(o)} + p^{o(e)}_{e(o)}(t)\rho_{\sigma(t)o(e)} \tag{50}$$

with

$$p^{e(o)}_{e(o)}(t) = \frac{1}{2}\,\frac{1+(-)\langle-\sigma(t)|\sigma(t)\rangle}{1+(-)\langle-\sigma_0|\sigma_0\rangle}\left(1+\frac{\langle-\sigma_0|\sigma_0\rangle}{\langle-\sigma(t)|\sigma(t)\rangle}\right), \tag{51}$$

$$p^{o(e)}_{e(o)}(t) = \frac{1}{2}\,\frac{1-(-)\langle-\sigma(t)|\sigma(t)\rangle}{1+(-)\langle-\sigma_0|\sigma_0\rangle}\left(1-\frac{\langle-\sigma_0|\sigma_0\rangle}{\langle-\sigma(t)|\sigma(t)\rangle}\right). \tag{52}$$

Observe that an initial even or odd superposition state evolves as a mixture of even and odd superposition states. Eq. (51) gives the probability of having a state whose parity is the same as that of the initial state, and Eq. (52) of opposite parity.

5. Determination of the master equation coefficients

Let us return to Eq. (10). If we call w_μ the exact eigenfrequencies of (1), we can write (remember that greek indices can assume the value 0, while latin indices do not)

$$a_\nu(t) = \sum_{\mu\sigma} U^*_{\mu\nu}U_{\mu\sigma}\mathrm{e}^{-\mathrm{i}w_\mu t}a_\sigma(0) = \sum_\sigma Z_{\nu\sigma}a_\sigma(0). \tag{53}$$

For the sake of convenience we write

$$\eta = Z_{00}, \quad \gamma_k = Z_{0k}, \quad \Delta_k = Z_{k0}, \quad \Gamma_{kl} = Z_{kl}. \tag{54}$$

Using (53) in Eqs. (17) and (18), we obtain the following expressions for η_k and γ_{kl}:

$$\eta_k = \frac{\Delta_k}{\eta}, \quad \gamma_{kl} = \Gamma_{kl} - \frac{\Delta_k\gamma_l}{\eta}. \tag{55}$$

From (19) and (20) we obtain

$$\lambda = -\sum_k c_k\,\mathrm{Im}(\eta_k),$$

$$\lambda' = -\sum_k c_k(2n_k(\beta)+1)\,\mathrm{Im}(\beta_k) = -\sum_k c_k(2n_k(\beta)+1)\,\mathrm{Im}\left(\sum_l \gamma_l\gamma^*_{kl}\right). \tag{56}$$

Since U is unitary we have

$$\sum_l \gamma_l\Gamma^*_{kl} = \sum_{\lambda\mu\nu} U^*_{\mu0}U_{\mu\lambda}U_{\nu k}U^*_{\nu\lambda}\mathrm{e}^{-\mathrm{i}(w_\mu-w_\nu)t} - \sum_{\mu\nu} U^*_{\mu0}U_{\mu0}U_{\nu k}U^*_{\nu0}\mathrm{e}^{-\mathrm{i}(w_\mu-w_\nu)t}$$
$$= \sum_{\mu\nu} U^*_{\mu0}U_{\nu k}\mathrm{e}^{-\mathrm{i}(w_\mu-w_\nu)t}\delta_{\mu\nu} - \eta\Delta^*_k = \sum_\mu U^*_{\mu0}U_{\mu k} - \eta\Delta^*_k = -\eta\Delta^*_k \tag{57}$$

and

$$\sum_l \gamma_l \gamma_l^* = \sum_{\lambda\mu\nu} U_{\mu 0}^* U_{\mu\lambda} U_{\nu 0} U_{\nu\lambda}^* \mathrm{e}^{-\mathrm{i}(w_\mu - w_\nu)t} - \sum_{\mu\nu} U_{\mu 0}^* U_{\mu 0} U_{\nu 0} U_{\nu 0}^* \mathrm{e}^{-\mathrm{i}(w_\mu - w_\nu)t}$$
$$= 1 - \eta\eta^* . \tag{58}$$

Using (57) and (58) we get

$$\beta_k = \sum_l \gamma_l \gamma_{kl}^* = \sum_l \gamma_l \Gamma_{kl}^* - \frac{\Delta_k^*}{\eta^*} \sum_l \gamma_l \gamma_l^* - \eta \Delta_k^* - \frac{\Delta_k^*}{\eta^*}(1 - \eta\eta^*) = -\eta_k^* . \tag{59}$$

Thus, we finally note that $\lambda'(t)$ can be expressed as a sum

$$\lambda'(t) = -\sum_k c_k (2n_k(\beta) + 1)\,\mathrm{Im}(\eta_k) = \lambda(t) - 2\sum_k c_k n_k(\beta)\,\mathrm{Im}(\eta_k)$$
$$= \lambda(t) + \varepsilon(t;\beta) . \tag{60}$$

Observe that $\lim_{\beta\to\infty}\varepsilon(t;\beta)=0$. This proves that indeed we have the equality $\lambda(t)=\lambda'(t)$ at zero temperature, as can be seen by looking at their definitions as given in Eq. (56).

We have seen that δ, λ and ε (or Ω, Λ and $\mathcal{N}$) are all we need to characterize the effect of the bath on the main oscillator. The Heisenberg equations of motion,

$$\dot{a} = -\mathrm{i}\omega_0 a - \mathrm{i}\sum_k c_k a_k , \tag{61}$$

$$\dot{a}_k = -\mathrm{i}\omega_k a_k - \mathrm{i}\, c_k a \tag{62}$$

can be solved in a number of ways. For example, an implicit method gives

$$a(t) = a(0)\mathrm{e}^{-\mathrm{i}\Omega(t) - \Lambda(t)} - \mathrm{i}\sum_k c_k a_k(0) \int_0^t \mathrm{d}\tau \mathrm{e}^{-\mathrm{i}\omega_k(t-\tau)} \mathrm{e}^{-\mathrm{i}\Omega(\tau) - \Lambda(\tau)} , \tag{63}$$

where $\eta(t)$ satisfies the integrodifferential equation

$$\dot{\eta} + \mathrm{i}\omega\eta + \int_0^t \mathrm{d}\tau \sum_k c_k^2 \mathrm{e}^{\mathrm{i}\omega_k(t-\tau)} \eta(\tau) = 0 , \tag{64}$$

subject to the initial condition $\eta_0 = 1$. Using the $\eta_k(t)$ implicitly defined in Eq. (19), and taking into account the equation satisfied by η, we obtain $\lambda + \mathrm{i}\delta = -\mathrm{i}\omega + \mathrm{d}(\ln\eta)/\mathrm{d}t$, or $\eta(t) = \exp(-\Lambda(t) - \mathrm{i}\Omega(t))$. We also find

$$\varepsilon(t) = \frac{\mathrm{e}^{-2\Lambda(t)}}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{2\Lambda(t)} \sum_k c_k^2 \left| \int_0^t \mathrm{d}\tau \mathrm{e}^{-\mathrm{i}\omega_k(t-\tau)} \mathrm{e}^{-\mathrm{i}\Omega(\tau) - \Lambda(\tau)} \right|^2 \eta_k(\beta) \right) . \tag{65}$$

Since (64) can be hard to solve, methods to obtain approximate solutions are welcome. For instance, we can expand in powers of c_k to second order, $\alpha(t)=\exp(k_0+ck_1+c^2k_2)$, to obtain

$$-\mathrm{i}\Omega(t) - \Lambda(t) = -\mathrm{i}\omega_0 t - \sum_k c_k^2 \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \mathrm{e}^{-\mathrm{i}(\omega_k - \omega_0)t_2} . \tag{66}$$

We expect (66) to be valid for small c_k. It can be shown that this is the time-dependent Born–Markov approximation, and that it is valid also for "strong" coupling and short times [15].

6. Experimental characterization

We observe that the calculation of the mean energy, and of the entropy of the above examples of initial conditions only require the knowledge of $\Lambda(t)$. However, if for example one wants to measure the Wigner function of any field density matrix, one would need to know $\Omega(t)$ as well. To determine this function one takes advantage of the experimental setup to measure the Wigner function [16], in which a two-level atom is prepared in the excited state $|e\rangle$, sent through an array of two low-Q cavities R1 and R2 and one high-Q cavity C between them, and is detected eventually. The field in C is displaced by the operator $D(\alpha)=\exp(\alpha a^\dagger - \alpha^* a)$, by a microwave source connected to it. R1 and R2 behave as "rotation" operators in the Hilbert space of atomic states, $|e\rangle \to (|e\rangle + e^{i\xi}|g\rangle)/\sqrt{2}$, and $|g\rangle \to (-e^{-i\xi}|e\rangle + |g\rangle)/\sqrt{2}$, with $\xi = 0$ in R1 and $\xi = \pi/2$ in R2. The dispersive atom–field interaction in C produces entanglement: The field component associated with $|e\rangle$ undergoes a phase shift implemented by $\exp(i\pi(a^\dagger a + 1)/2)$, while the one associated with $|e\rangle$ a phase shift $\exp(-i\pi a^\dagger a/2)$. It was shown in Ref. [16] that for this experimental arrangement

$$\Delta P = P_e - P_g = W(-\alpha, -\alpha^*, t)/2 , \tag{67}$$

where $P_e(P_g)$ is the probability to detect the probe atom in the upper (lower) state $|e\rangle(|g\rangle)$, and $W(-\alpha, -\alpha^*, t)$ is the value of the Wigner function corresponding to the field in the high-Q cavity at the time t the probe atom exits this cavity. Note that, for the sake of convenience the normalization of W has been changed by a factor of π. The Wigner function of the coherent state (44) is

$$W(\alpha, \alpha^*, t) = 2e^{-2(\sigma(t)-\alpha)(\sigma(t)-\alpha)^*} . \tag{68}$$

Note that $W(\alpha, \alpha^*, t)$ presents a maximum at $\alpha = \sigma(t) = \sigma_0 \exp(-\Lambda(t) - i\Omega(t))$. Since Λ can be determined from a photocounting experiment, we just need to measure $\Omega(t)$. We choose $\alpha = \alpha(t) = \sigma_0 \exp(-\Lambda(t)) \exp(-i\Phi(t))$, and adjust $\Phi(t)$ to maximize ΔP. If $\Phi(t) = \Omega(t) \bmod (2\pi)$, then we obtain $\Delta P = 1$, otherwise, it will be smaller than one, $\Delta P = \exp(-8\sigma_0^2 \exp(-2\Lambda) \sin^2((\Omega - \Phi)/2))$. Choosing different exit times t we can map $\Omega(t)$. Note that what we do, in fact, is find the right $\alpha(t)$ that brings the field at the cavity C to the vacuum. If we succeed in doing so the probe atom is rotated in R1, its state becomes a superposition of upper and lower states, it then flies free until R2 where the rotation is reversed, and reaches the detectors again in the upper state.

7. Concluding remarks

In conclusion, we have shown that the reduced density of a harmonic oscillator linearly coupled to its environment with an RWA-type of coupling obeys an equation which has precisely the same form of the usual Born–Markov equation. If the optical cavities can be modelled through Hamiltonian (1), then only *three*, *experimentally measurable* real functions are necessary to characterize their behavior: the mean photon number of an initial ground state, the instantaneous frequency and the rate of change of the number of excitations. Current experimental arrangements can assess these parameters.

References

[1] M. Brune, E. Hagley, J. Dreyer, X. Maître, A. Maali, C. Wunderlich, J.M. Raimond, S. Haroche, Phys. Rev. Lett. 77 (1996) 4887.
[2] M. Brune, S. Haroche, V. Lefevre, J.M. Raimond, N. Zagury, Phys. Rev. Lett. 65 (1990) 976;
M. Brune, S. Haroche, J.M. Raimond, L. Davidovich, N. Zagury, Phys. Rev. A 45 (1992) 5193;
V. Buzek, H. Moya-Cessa, P.L. Knight, S.J.D. Phoenix, Phys. Rev. A 45 (1992) 8190;
S. Haroche, J.P. Raimond, In: J.H. Eberly, L. Mandel, E. Wolf (Eds.), Coherence and Quantum Optics, Vol. VII, Plenum Press, New York, 1996, p. 83;
L. Davidovich, M. Brune, J.M. Raymond, S. Haroche, Phys. Rev. A 53 (1996) 1295.
[3] B.L. Hu, J.P. Paz, T. Yu, Phys. Rev. D 45 (1992) 2843.
[4] J. Anglin, S. Habib, Mod. Phys. Lett. A 11 (1996) 2655.
[5] J.J. Halliwell, T. Yu, Phys. Rev. D 53 (1996) 2012.
[6] G.M. Moy, J.J. Hope, C.M. Savage, Phys. Rev. A 59 (1999) 667.
[7] S. Bay, P. Lambropoulos, K. Molmer, Phys. Rev. Lett. 79 (1997) 14.
[8] W.H. Louisell, Quantum Statistical Properties of Radiation, Wiley, New York, 1973.
[9] H.E. Borges, Ph.D. Thesis, Universidade Federal de Minas Gerais, 1996.
[10] M.C. Nemes, A.F.R. de Toledo Piza, Physica A 137 (1986) 367.
[11] N.G. van Kampen, J. Stat. Phys. 78 (1995) 299;
K.M. Fonseca Romero, M.C. Nemes, Phys. Lett. A 235 (1997) 432–437.
[12] J.J. Hope, Phys. Rev. A 55 (1997) 2531.
[13] H.F. Arnoldus, JOSA B 13 (1996) 1099.
[14] V. Buzek, A. Vidiella-Barranco, P.L. Knight, Phys. Rev. A 45 (1992) 6570.
[15] W.D. Jose, M.C. Nemes, A.F.R. de Toledo Piza, Physica A 283 (2000) 359.
[16] L.G. Lutterbach, L. Davidovich, Phys. Rev. Lett. 78 (1997) 2547.