Quantum decoherence without damping

K.M. Fonseca Romero a,1, M.C. Nemes b,2

a Instituto de Física, Departamento de Física Matemática, Universidade de São Paulo, C.P. 20516, 01317-970 São Paulo, S.P., Brazil
b Departamento de Física, ICEX, Universidade Federal de Minas Gerais, C.P. 702, 30161-970 Belo Horizonte, M.G., Brazil

Received 11 December 1996; revised manuscript received 6 May 1997; accepted for publication 12 August 1997
Communicated by P.R. Holland

Abstract

An exactly soluble model leading to decoherence is studied. The effect of the environment (subohmic, ohmic and supraohmic) and of the temperature (zero or finite) is presented. The accuracy of the Born–Markov approximation and of the semigroup evolution equation is discussed. © 1997 Published by Elsevier Science B.V.

PACS: 05.40.+j; 42.50.Lc

The description of a selected set of observables of a given quantum system has become an important research topic in different areas of physics, such as cosmology [1], quantum optics [2], condensed matter [3], particle physics [4] and the fundamentals of the quantum measurement process [5]. The problem can be stated very generally by considering several interacting subsystems and deriving the effective dynamics of the system of interest. The exact effective dynamics of a quantum subsystem, part of an isolated system, has been derived in Ref. [6]. It is shown to be non-linear and non-Markovian in general. From the strictly formal point of view one can learn that the effects of quantum correlations on the effective dynamics of interest are two-fold: unitary-type correlation effects which will give rise to (time-dependent) corrections to the mean field subsystem dynamics and corrections of non-unitary type which give rise to dissipative as well as decoherence effects. These two effects, dissipation and decoherence, although having the same physical origin, are quite distinct phenomena as discussed for example in Refs. [7,8] (Ref. [7] examines the relevance of the spectral density of the bath oscillators on the decoherence process in a model leading to both decoherence and dissipation). Given the highly intricate and complex character of the exact dynamical evolution of quantum subsystems, and the lack of sound mathematical statements about its behaviour, the quantum semigroup dynamics is usually postulated to investigate the problem. Recently the adequacy of such hypothesis has been investigated in the literature [9].

The purpose of the present contribution is also in the same spirit of Ref. [9]. We set up the most general model describing a system of interest \( S \) characterized by a generic Hermitian operator \( \hat{A} \), interacting with a reservoir of harmonic oscillators. The coupling is assumed to be a function of \( \hat{A} \) so that the free part of the \( S \) system's dynamics commutes with the full Hamiltonian. An immediate consequence of this assumption

---

* Partially supported by CNPq, Brazil.
1 On leave from CIF, Colombia.
2 E-mail: carolina@fisica.ufmg.br.

0375-9601/97/$17.00 © 1997 Published by Elsevier Science B.V. All rights reserved.

PII S0375-9601(97)00659-2
is that there will be no dissipative process in the dynamics, although the decoherence process as it will be seen, is fully active. In the present work the validity of the semigroup assumptions is not investigated in detail, since this has been done before, but the focus is on the physics of the decoherence process in this exactly soluble model. It is also investigated how much of this physics is absent due to the Markovian approximation. The first remarkable effect found is that the properties of the decoherence process for short and large times are intimately connected to those of the bath distribution frequencies. It is found that one can have asymptotically partial decoherence or total decoherence with different asymptotic behaviour in supraohmic, ohmic or subohmic regimes, being exponential only in the ohmic case. Moreover, it is found that the zero temperature limit is qualitatively very different from the finite temperature case. The usual Born–Markov approximation, however, is unable to predict partial decoherence. In this case the decoherence is either instantaneous (in the subohmic regime), exponential (in the ohmic case) or zero for the supraohmic environment. Also for $T = 0$ no decoherence at all is found for the usual semigroups assumptions. The short time behaviour of the idempotency defect [10] is very different. The exact calculations show a correspondence with the result predicted from the short time expansion of Ref. [10], i.e. proportional to $t^2$ (departure from this general behaviour is found only when $T = 0$ and the bath spectral density falls slowly to zero at infinity). In the Markovian case, whenever there is decoherence, i.e. the ohmic regime, the short time behaviour is linear in time.

The model Hamiltonian we consider, given by

$$H = h(A) + h f(A) \sum c_i(a_i^\dagger + a_i) + h g(A) \sum i d_i(a_i^\dagger - a_i) + h \sum \omega_i a_i^\dagger a_i,$$

(1)

describes the interaction of a system $S$, whose free Hamiltonian is $h(A)$, with a collection of oscillators, through an interaction which conserves the system’s observable $A$. The creation and annihilation operators $a_i^\dagger$ and $a_i$ satisfy the usual boson commutation relations. Hamiltonians of the type we consider here are expected to be approximately valid when the characteristic decoherence time of a system $S$, which depends on the initial state of the system, is much smaller than

the thermal equilibrium relaxation time. Hamiltonians of the type proposed here are known to be (the most simple) models satisfying the conditions for a QND meter (with the collection of oscillators playing the role of the meter) [11].

We assume a factorized initial total density matrix given by

$$\rho(t = 0) = \rho_S(t = 0) \rho_B(t = 0)$$

$$= \sum |m\rangle \rho_S^{mn} \prod \frac{e^{-\beta h \omega_i} a_i^\dagger a_i}{Tr e^{-\beta h \omega_i} a_i^\dagger a_i} \langle n|,$$

(2)

where the system’s density matrix $\rho_S$ has been written in terms of a complete set of eigenvectors $|m\rangle$ of $A$, with eigenvalues $\omega_i$, and the ensemble of oscillators, with density matrix $\rho_B$, is considered to be in thermal equilibrium at inverse temperature $\beta$. Using Lie-algebraic techniques [12] we can compute the total density matrix at a later time $t$,

$$\rho(t) = e^{H t / i h} \rho(0) e^{-H t / i h}$$

$$= \mathcal{U}(t) \mathcal{V}(t) (\mathcal{D}(t) \rho_C(t)),$$

(3)

where $\mathcal{U}(t)$, $\mathcal{V}(t)$, $\mathcal{D}(t)$ and $\rho_C(t)$ are given below. Taking the trace over the bath degrees of freedom one obtains

$$\rho_S(t) = \mathcal{U}(t) \mathcal{V}(t) (\mathcal{D}(t) \rho_S(0)).$$

(4)

Eq. (4) and the analysis of the expressions for $\mathcal{U}(t)$, $\mathcal{V}(t)$ and $\mathcal{D}(t)$, which only depend on the system’s observable $A$, show that, from the system’s viewpoint, $\mathcal{U}$ is an explicitly Hamiltonian unitary contribution, $\mathcal{V}$ a non-explicitly Hamiltonian unitary contribution, and $\mathcal{D}$ a non-unitary contribution.

$\mathcal{U}$ is given by

$$\mathcal{U}(t) = \exp \left( \frac{1}{i h} \int_0^t d\tau \{ k(A; \tau), \ldots \} \right),$$

(5)

where we employ the same dot convention as Ref. [13]. The “renormalized Hamiltonian” $k(A, t)$ is

$$k(A; t) = h(A) - f^2(A) R_f(t) - g^2(A) R_g(t),$$

(6)

$$R_f(t) = \hbar \sum c_i^2 \frac{1}{\omega_i} \left[ 1 - \cos(\omega_i t) \right]$$

$$\rightarrow \hbar \int_0^\infty d\omega \, D_f(\omega) \left[ 1 - \cos(\omega t) \right].$$

(7)
The arrow shows the expression that is obtained after having taken the limit to the continuum. The spectral density $D_f(\omega)$ comprises the effect of the coupling $(c(\omega))$, the frequency $(\omega)$ and the density of oscillators $(n(\omega))$, $D_f(\omega) = c(\omega)n(\omega)/\omega$ [7]. Replacing $c_i(D_f(\omega))$ by $d_i(D_g(\omega))$ in the expression for $R_g(t)$ one obtains the corresponding expression for $R_g(t)$. The same observation applies to other quantities below. If we consider only renormalized Hamiltonians with a lower bound for all times, some restrictions must be imposed both on the functional form of $f$ and $g$, and on $R(t)$, which depends on the coupling constants and the spectral density of the bath. In particular, we demand $R(t)$ to be a bounded function. This is easy to implement noting that $R$ satisfies the inequality $0 < R(t) < 2R(\infty)$ for $t > 0$.

The superoperator $\mathcal{V}$ is given by

$$
\mathcal{V}(t) = \exp \left( -2i \int_0^t d\tau \, m(\tau) \right) \left( [f(A),g(A) - g(A)\cdot f(A)] \right),
$$

$$
m(t) = \sum c_i d_i \sin(\omega_i t)
- \int d\omega \sqrt{D_f(\omega)} D_g(\omega)
\times \text{sgn}(c(\omega)d(\omega)) \sin(\omega t).
$$

Note that $m$ satisfies the inequality $|m(t)|^2 < R_f(\infty) R_g(\infty)$ and $\lim_{t\to\infty} m(t) = 0$.

It is worth remarking that $\mathcal{V}$ (i) conserves the trace $\text{Tr}(\mathcal{V}(t) \rho) = \text{Tr}(\rho)$; (ii) does not modify the Hermiticity of $\rho$, $(\mathcal{V}(t) \rho_0)^\dagger = \mathcal{V}(t) \rho_0$ whenever $\rho_0^\dagger = \rho_0$, and (iii) conserves the degree of mixture, $\text{Tr}(\mathcal{V} \rho_0)^2 = \text{Tr} \rho_0^2$. Therefore one should be able to find a Hamiltonian $K$ such that $i\hbar \mathcal{V} = [K, \mathcal{V}]$ [14]. We did not find the appropriate $K$ in the general case. However, if $\rho$ is a pure state for all $t$, $\rho = |\psi_t\rangle \langle \psi_t|$, then $K = 2\hbar m(t) (f(\psi_t)|g(\psi_t) - g(\psi_t)|f(\psi_t))$, a nonlinear Hamiltonian.

Moreover, for the non-unitary operator $\mathcal{D}$ one finds

$$
\mathcal{D} = \exp[-\lambda_f(t)(\dot{f}^2 - 2f \cdot \dot{f} + f^2)
- \lambda_g(t)(\dot{g}^2 - 2g \cdot \dot{g} + g^2)],
$$

$$
\lambda_f = \sum \frac{c_i^2 [1 - \cos(\omega_i t)] \coth(\frac{1}{2} B\hbar \omega_i)}{\omega_i^2}
- \int d\omega D_f(\omega) [1 - \cos(\omega t)] \coth(\frac{1}{2} B\hbar \omega).
$$

Calculating the linear entropy (or idempotency defect) one finds

$$
\delta(t) = 1 - \text{Tr} \rho_0^2
= 1 - \sum_r |\rho_g^{(r)}|^2 \exp\{-2\lambda_f(t) [f(A_m) - f(A_n)]^2
- 2\lambda_g(t) [g(A_m) - g(A_n)]^2\}.
$$

Eq. (12) substantiates the claim that $\mathcal{D}$ is the only non-unitary effect and leads to decoherence. The decoherence coefficient $\lambda$ can be split as $\lambda_f(t) = \lambda_f^0(t) + \lambda_f^\beta(t)$, with

$$
\lambda_f^0(t) = \int_0^\infty \frac{D_f(\omega)}{\omega} \frac{1 - \cos(\omega t)}{\omega^2} d\omega,
$$

$$
\lambda_f^\beta(t) = 2 \int_0^\infty \frac{D_f(\omega)}{\omega} \frac{1 - \cos(\omega t)}{\omega^2} d\omega,
$$

where the zero temperature contribution has been isolated from the finite temperature one. This separation is important, because their behaviour is qualitatively different (in particular, the behaviour of the expressions under the integral sign, in the neighbourhood of $\omega = 0$, is different). Note that $\lambda$ satisfies the inequality $0 < \lambda(t) < 2\lambda(\infty)$. $\lambda^0$ and $\lambda^\beta$ satisfy similar relations.

From the positivity of $\lambda(t)$ follows the inequality $\delta(t) \leq \delta(0)$. If additionally $\lambda(t) \geq 0$ it follows that $\delta(t') \geq \delta(t)$ for all $t' \geq t \geq 0$ no matter what the initial $\rho_0$ is. Other models, which can be put in the Lindblad form, require to have a pure state $\rho(t)$ in addition to $\lambda(t) > 0$ to obtain a growing idempotency defect. This can have a physical origin in a model leading to thermalization: we expect a system initially at a temperature higher than that of the bath, to decrease its temperature, decreasing its idempotency defect.

Finally, the “correlated density operator” $\rho_C(t)$ given by
\[ \rho_{C}(t) = \sum |n \rangle \rho_{S}^{m n} \times \prod \frac{\exp[- \beta \hbar \omega a_{m n}^{\dagger}(t)a_{m n}(t)]}{\text{Tr} e^{- \beta \hbar \omega a_{m n}^{\dagger} a_{m n}}} |m \rangle, \]  
\[ a_{m n}(t) = a_{i} + s_{i}(1 - e^{- \beta \hbar \omega_{i}}) \]
\[ - \frac{(s_{i} - s_{i}) (1 - e^{- \beta \hbar \omega_{i}})}{1 - e^{- \beta \hbar \omega_{i}}}, \]
\[ a_{m n}^{\dagger}(t) = a_{i}^{\dagger} + s_{i}^{*}(1 - e^{\beta \hbar \omega_{i}}) \]
\[ - \frac{(s_{i}^{*} - s_{i}^{*}) (1 - e^{\beta \hbar \omega_{i}})}{1 - e^{- \beta \hbar \omega_{i}}}, \]
\[ s_{i} = \frac{c_{i}}{\omega_{i}} f(A_{m}) + i \frac{d_{i}}{\omega_{i}} g(A_{m}), \]

Eq. (20) shows that under some circumstances the bath effectively acts as some kind of measurement device.

From expressions (7), (9), (13) and (14) it is easily seen that the short (large) time behaviour depends on the high (low) frequencies of the spectral density. In particular, for bath spectra decaying faster than any polynomial, \( R \) and \( \lambda \) admit nice series expansions,
\[ R(t) = \sum_{n=1}^{\infty} (-1)^{n+1} a_{n}^{2} t^{2n}, \]
\[ \lambda(t) = \sum_{n=1}^{\infty} (-1)^{n+1} b_{n}^{2} t^{2n}. \]

Two different types of spectral densities for the oscillators are considered, \( D_{1}(\omega) = (\omega/\omega_{c})^{2} \theta(\omega - \omega_{c}) \), and the more general and widely used one \( D_{2}(\omega) = (\omega/\omega_{c})^{\nu - 1} \exp(-\omega/\omega_{c}) \) for \( \nu > 0 \), where \( \omega_{c} \) stands for the Debye frequency and \( \nu \) describes the type of environment, supraohmic if \( \nu > 1 \), ohmic if \( \nu = 1 \) and subohmic if \( 0 < \nu_{c} < 1 \). While the latter is good for testing differences in the large time behaviour, the former is good to show short time behaviour not satisfying (22).

Some results for \( R \) and \( \lambda \) are presented in Tables 1–3, where the following conventions has been adopted: \( E = \hbar \omega_{c}, \tau = \omega_{c} t, \beta' = \hbar \omega_{c} \beta, C \) is the Euler-Mascheroni constant, Im denotes the imaginary part, \( \xi \) is the generalized zeta function, si the integral sine, ci the integral cosine and \( \xi \) and \( \theta \) are real numbers such that \( \xi > 1 \) and \( 0 < \theta < 1 \). Results for \( R(t) \) appear in Table 1, and Tables 2 and 3 contain the decoherence properties separated for zero temperature (\( \lambda(0)(\tau) \)) and finite temperature (\( \lambda(\beta)(\tau) \)) in all regimes. We do not make explicit calculations for \( m(t) \). The short and asymptotic behaviour is shown for all cases. While in the case of \( D_{2}(\omega) \) the short time behaviour always goes as \( t^{2} \), the slowly decaying \( D_{1}(\omega) \) gives rise to a linear term in the expansion of \( R \) and to a non-analytic
term \( t^2 \ln t \) in the expansion of \( \lambda^0 \). \( \lambda^0 \) always goes as \( \tau^2 \) for short \( \tau \). On the other hand, while the long time decoherence properties of \( D_1(\omega) \) are the same whether at zero or finite temperature, those of \( D_2(\omega) \) depend both on the presence (or absence) of temperature and on the value of the parameter \( \nu \).

In situations in which the system of interest interacts with a large environment it is common to use a master equation approach. It is easy to see that from (4) follows a generalized master equation with time-dependent coefficients for \( \rho \).
Table 4  
Renormalization and decoherence coefficients in the BMA

<table>
<thead>
<tr>
<th>$D(\omega)$</th>
<th>$R(\tau)$</th>
<th>$m(\tau)$</th>
<th>$\lambda^0(\tau)$</th>
<th>$\lambda^0(\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>$E$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$D_2$</td>
<td>$f(\nu)E$</td>
<td>0</td>
<td>0</td>
<td>$\nu &gt; 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\pi/\beta'$</td>
<td>$\nu = 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\infty$</td>
<td>$0 &lt; \nu &lt; 1$</td>
</tr>
</tbody>
</table>

$$\rho_S(t) = \left( \frac{1}{i\hbar} [k,.] - 2i m (f^g - g.f) \right.$$  
$$- \lambda_f (f^2 - 2 f.f + .f^2)$$  
$$- \lambda_g (g^2 - 2 g.g + .g^2) \right) \rho_S(t). \quad (23)$$

As long as the second term of the r.h.s. of the above equation should correspond to some Hamiltonian $K$ as pointed out before, (23) can be seen as an obvious generalization of the Lindblad form allowing for time-dependent generators. If $\lambda_f$ and $\lambda_g$ are non-negative, (23) is in Lindblad form for every fixed positive time and we expect the von Neumann conditions to hold [13]. If, moreover, either $\lambda_f$ or $\lambda_g$ is zero, or one is equal to the other, a simple rescaling of the time variable will lead (23) to Lindblad form, with time-dependent Hamiltonian, over time intervals in which $\lambda$ is strictly positive.

Finally we solve the same problem using the Born–Markov approximation (BMA). The results are shown in Table 4.

Notice that the MBA does not allow for non-explicitly Hamiltonian unitary corrections. Also, as discussed before, decoherence is either absent or total. It is particularly “dangerous” using the MBA for zero temperature, where the exact result shows decoherence, whereas the approximation does not allow for any decoherence. We conclude by noting that although in this simple case it is possible to write down an equation for $\rho$ in a generalized Lindblad form, the physical content might be quite different since transient effects (usually neglected) may be rather important as far as the decoherence process is concerned.

References


