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Sensitivity to initial conditions in quantum dynamics: an analytical semiclassical expansion

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Abstract

Given a quantum Hamiltonian of point particles and angular momenta, we give a procedure to define a corresponding semiclassical dynamics with essentially classical content, around which the quantum dynamics can be expanded. The modulus of the quantum overlap of coherent states, evolved with the semiclassical Hamiltonian, naturally introduces a classical distance between classical phase points. Using this fact we analytically show that the time rate of change (trc) of two neighboring classical trajectories is directly proportional to the trc of quantum correlations. Coherence loss and nonlocality effects appear as corrections to semiclassical dynamics and we show that they can be perturbatively given in terms of classical trajectories and generalized actions. We apply the results to the nonintegrable (classically chaotic) version of the N-atom Jaynes–Cummings model.

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1. Introduction

Ever since the conception of quantum mechanics the classical limit has been a matter of much debate due to the profound contrasting differences between the classical and quantum descriptions of the world. Although the difficulties in building a bridge between quantum and classical mechanics are well known, we start by reviewing the ones which are of relevance for the present contribution. As far as kinematical differences are concerned, already at the level of a point particle, striking differences appear. While the definition of the state of a classical particle is of local char-

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acter and given by a point in phase space, the quantum counterpart of the definition of a particle state is given by a vector in Hilbert space which cannot simultaneously be ascribed a well defined value for position and momentum. The closer one can get to the classical situation are minimum uncertainty wave packets. Quantum states are therefore usually nonlocal. Also, the linear character of the Hilbert space has the immediate consequence that superposition of (minimum uncertainty) states are also possible states and, in fact, constitute the vast majority of allowed quantum states. The situation gets even cloudier when two degrees of freedom are involved: classically one can always describe a two particle state in terms of the coordinates and momenta of each one of them. Quantum mechanically, however, this situation only holds if the two particles, initially in a factorized state, do not interact. The Hilbert space structure allows for states which cannot be written as a direct product of vectors in the individual Hilbert spaces of each degree of freedom. This essentially quantum property is usually named entanglement. Much investigation and progress both on the theoretical and experimental sides have been achieved recently [1].

From the dynamical point of view one of the essential differences has given rise to an important research area nowadays: classical chaos, a phenomenon whose root lies on the nonlinearity of Newton's equation. The relevant question here is how to identify the quantum counterpart of classical chaos. A major step in this direction was given by Bohigas and collaborators who conjectured that spectral properties of integrable and nonintegrable systems should be very different [2]. Thereafter many numerical investigations confirmed such conjecture and a few exceptions were found. From the analytical point of view a most relevant formula was derived by Gutzwiller connecting level densities of very general quantum systems with classical periodic orbits and their actions [3]. In what concerns the connections between classical and quantum dynamics, quantum corrections, in the sense of a $\hbar$ expansion, have been studied for nonautonomous single mode bosonic systems [4] and for nonresonant N-atom Jaynes–Cummings model [5]. The growth of the quantum corrections is identified with a transition from "regular" to "stochastic" behavior in these systems. Also it has recently been proposed by Zurek that the rate of entropy production can be used as an intrinsically quantum test of the chaotic vs. regular nature of the evolution [6]. Several numerical tests of this conjecture can also be found [7,8]. Anyway, analytic results in this context are scarce. As a contribution of the present Letter to this theme, we give a recipe to construct a semiclassical dynamics (SCD) for systems composed of both point particle and angular momenta, for initial coherent states, and show that the quantum corrections to the SCD are perturbatively given in terms of the classical trajectories and generalized actions. Since the SCD depends on the initial state, the (semiclassical) overlap between initially close coherent states, in contradistinction with the exact dynamics, varies in time. Thus, the high sensitivity to the initial conditions of classically chaotic systems is reflected in the change of the semiclassical overlap modulus, a natural measure of distance between states, and in the production rate of quantum corrections to the overlap.

Our Letter is divided as follows. In Sections 2 and 3 we present a procedure to find the semiclassical Hamiltonian (SCH), and a perturbative scheme around it, respectively. In Section 4 the overlap sensitivity to initial conditions is discussed. Finally, in Section 5, the maser is used as an example.

2. Zeroth order approximation: semiclassical approximation (SCA) and classical limit

The general strategy is as follows. We consider a system of m point particles and n angular momenta, $\boldsymbol{H}$, and a product state composed of m coherent states and n spin coherent states, as initial states. Linear combinations are not allowed as initial conditions (this circumvents problems with the superposition principle). The expectation value of the system Hamiltonian in this product state corresponds to the classical Hamiltonian. Then, we define the SCH $\boldsymbol{H}_{\mathrm{sc}}$, such that (i) preserve the product of coherent states, and (ii) generate Heisenberg equations that are similar to the classical equations and, when averaged, coincide with them.

The procedure to find the semiclassical dynamics, that we expose here, is also viable for initial quantum states that are localized, even if not strictly coherent. The semiclassical dynamics will not be, however, as easy as in the present case, and the corrections will look different. For initial delocalized and/or entangled state the present approach is unsuitable, because it is aimed at the identification of the separation of quantum and classical dynamics, provided that the initial state is to some extent "classical".

For the sake of clearness we shortly review some facts concerning coherent states.

The minimum uncertainty coherent states $|z\rangle$ are defined by $|z(t)\rangle = \mathcal{D}(z(t))|R\rangle$ with

$$\mathcal{D}\big(z(t)\big) = \begin{cases} \exp\big[z(t)\boldsymbol{a}^\dagger - z^*(t)\boldsymbol{a}\big], \\ \exp\big[\frac{\operatorname{atan}|z(t)|}{|z(t)|}\big(z(t)\boldsymbol{J}_+ - z^*(t)\boldsymbol{J}_-\big)\big], \end{cases} \quad (1)$$

for point particles and angular momenta (spins) respectively, where the operators $\boldsymbol{a}^\dagger$, $\boldsymbol{a}$ are the usual creation and destruction operators, and $\boldsymbol{J}_+$, $\boldsymbol{J}_-$ the angular momentum ladder operators. The minimum uncertainty property of the so-called fiducial state $|R\rangle$, the Fock state $|0\rangle$ for the point particle and the $\boldsymbol{J}_z$ eigenstate $|J, -J\rangle$ for the angular momentum, is inherited to the coherent states. The point particle and the angular momentum operators obey the following commutation relations

$$\big[\boldsymbol{a}, \boldsymbol{a}^\dagger\big] = \mathbf{1}, \qquad [\boldsymbol{J}_z, \boldsymbol{J}_\pm] = \pm\boldsymbol{J}_\pm,$$
$$[\boldsymbol{J}_+, \boldsymbol{J}_-] = 2\boldsymbol{J}_z. \quad (2)$$

If the Hamiltonian is of the form

$$\boldsymbol{H}_{\mathrm{HO}}(t) = a_0(t)\boldsymbol{a}^\dagger\boldsymbol{a} + a_+(t)\boldsymbol{a}^\dagger + a_-(t)\boldsymbol{a}, \quad (3)$$
$$\boldsymbol{H}_{\mathrm{RM}}(t) = a_0(t)\boldsymbol{J}_z + a_+(t)\boldsymbol{J}_+ + a_-(t)\boldsymbol{J}_-, \quad (4)$$

respectively, for a point particle, and an angular momentum, an initial coherent state remains coherent through evolution, besides a phase factor. In (3) and (4) the subindices HO and RM correspond to typical representatives: the harmonic oscillator and rotating molecule. At this point it is important to mention that the harmonic oscillator Hamiltonian can be generalized with little effort, to include terms in $\boldsymbol{a}^2$, $(\boldsymbol{a}^\dagger)^2$ but the extra terms give rise to nonlocality effects (squeezing dynamics) already at this lowest order which we would like to avoid.

We say that a Hamiltonian is linear if it is linear in the operators $\boldsymbol{a}^\dagger\boldsymbol{a}$, $\boldsymbol{a}^\dagger$, $\boldsymbol{a}$, $\boldsymbol{J}_+$, $\boldsymbol{J}_-$, and $\boldsymbol{J}_z$, which we call the semiclassical generators; otherwise we refer to it as a nonlinear Hamiltonian. Even if we choose a coherent state as initial condition, if the Hamiltonian is nonlinear, the dynamics will modify it and the state will no longer be coherent. We expose our method by considering a nonlinear Hamiltonian for a point particle, for an angular momentum, and for a system composed of two degrees of freedom, for example, one particle and one angular momentum. The generalization to several degrees of freedom with arbitrary nonlinearities is straightforward.

2.1. Nonlinear point particle Hamiltonian

The classical Hamiltonian $\mathcal{H}(z, z^*)$ corresponding to a quantum Hamiltonian $\boldsymbol{H}$,

$$\mathcal{H}\big(z, z^*\big) = \langle z|\boldsymbol{H}\big(\boldsymbol{a}, \boldsymbol{a}^\dagger, \boldsymbol{a}^\dagger\boldsymbol{a}\big)|z\rangle, \quad (5)$$

can be written as a function of the expectation values z, z^*, and $|z|^2$ of the semiclassical generators $\boldsymbol{a}$, $\boldsymbol{a}^\dagger$, $\boldsymbol{a}^\dagger\boldsymbol{a}$ in the coherent state $|z\rangle$,

$$\mathcal{H}\big(z, z^*\big) = \mathcal{K}\big(z, z^*, |z|^2\big). \quad (6)$$

The classical Hamiltonian $\mathcal{K}$ is unique if the quantum Hamiltonian admit a Taylor expansion in the semiclassical generators, as we implicitly assume. The usual classical equation of motion

$$\frac{dz}{dt} = \frac{1}{i\hbar}\frac{\partial\mathcal{H}(z, z^*)}{\partial z^*}, \quad (7)$$

in terms of the classical Hamiltonian $\mathcal{H}$, can be recast as

$$\frac{dz}{dt} = \frac{1}{i\hbar}\left(\frac{\partial\mathcal{K}(z, z^*, |z|^2)}{\partial z^*} + \frac{\partial\mathcal{K}(z, z^*, |z|^2)}{\partial|z|^2}z\right) \quad (8)$$

in terms of the Hamiltonian $\mathcal{K}$. We stress that the classical and the quantum evolution give different values for the observables in the sense that

$$z(t) \neq \big\langle z(0)\big|\boldsymbol{a}(t)\big|z(0)\big\rangle. \quad (9)$$

However, if we compare the classical dynamics with the one provided by the *semiclassical* Hamiltonian $\boldsymbol{H}_{\mathrm{HO}}(t)$,

$$\frac{d\boldsymbol{a}}{dt} = \frac{1}{i\hbar}\big(a_+(t) + a_0(t)\boldsymbol{a}\big), \quad (10)$$

we see that they have the same structure. Analogous similarities can be found between the semiclassical equations for $\boldsymbol{a}$, $\boldsymbol{a}^\dagger\boldsymbol{a}$ and the classical equations for z^*, $|z|^2$, respectively. Moreover, the averaged semiclassical equations of motion coincide with the classical ones if the SCH is chosen as

$$\boldsymbol{H}_{\mathrm{HO}}(t)=\frac{\partial\mathcal{K}}{\partial|z|^2}\boldsymbol{a}^\dagger\boldsymbol{a}+\frac{\partial\mathcal{K}}{\partial z^*}\boldsymbol{a}^\dagger+\frac{\partial\mathcal{K}}{\partial z}\boldsymbol{a}. \tag{11}$$

2.2. Nonlinear angular momentum Hamiltonian

As in the previous subsection, the classical Hamiltonian $\mathcal{H}(z,z^*)$ is the expectation value of the quantum Hamiltonian $\boldsymbol{H}$, in the corresponding *spin* coherent states, and can be written in terms of the expectation values of the semiclassical generators,

$$\langle z|\boldsymbol{J}_{-(+)}|z\rangle=2J\frac{z^{(*)}}{1+|z|^2}=\mathcal{J}_{-(+)},$$
$$\langle z|\boldsymbol{J}_z|z\rangle=-J\frac{1-|z|^2}{1+|z|^2}=\mathcal{J}_z. \tag{12}$$

The classical equation of motion for z,

$$\frac{dz}{dt}=\frac{1}{i\hbar}\frac{1+|z|^2}{2J}\frac{\partial\mathcal{H}(z,z^*)}{\partial z^*}, \tag{13}$$

and the equation for z^*, can be used to find the dynamics for $\mathcal{J}_{-(+)}$ and $\mathcal{J}_z$. As before, we can define a Hamiltonian $\mathcal{K}(\mathcal{J}_-,\mathcal{J}_+,\mathcal{J}_z)=\mathcal{H}(z,z^*)$. The equation of motion for $\mathcal{J}_z$, which can finally be written as

$$\frac{d\mathcal{J}_z}{dt}=\frac{1}{i\hbar}\left(\frac{\partial\mathcal{K}}{\partial\mathcal{J}_+}\mathcal{J}_+-\frac{\partial\mathcal{K}}{\partial\mathcal{J}_-}\mathcal{J}_-\right) \tag{14}$$

exhibits a structure similar to equation of motion for $\boldsymbol{J}_z$, provided by the SCH $\boldsymbol{H}_{\mathrm{RM}}(t)$,

$$\frac{d\boldsymbol{J}_z}{dt}=\frac{1}{i\hbar}\big(a_+(t)\boldsymbol{J}_+-a_-(t)\boldsymbol{J}_-\big). \tag{15}$$

Analogous similarities can be found between the semiclassical equations for $\boldsymbol{J}_\pm$, and the classical equations for $\mathcal{J}_\pm$. Moreover, the averaged semiclassical equations of motion coincide with the classical ones if the SCH is chosen as

$$\boldsymbol{H}_{\mathrm{RM}}(t)=\frac{\partial\mathcal{K}}{\partial\mathcal{J}_z}\boldsymbol{J}_z+\frac{\partial\mathcal{K}}{\partial\mathcal{J}_+}\boldsymbol{J}_++\frac{\partial\mathcal{K}}{\partial\mathcal{J}_-}\boldsymbol{J}_-. \tag{16}$$

This form is adequate for the point of view of the structure. However, for the sole purpose of calculating the SCH, it is easier to use the expression

$$\boldsymbol{H}_{\mathrm{RM}}(t)=\frac{1+|z|^2}{2J}\left(\left(\frac{\partial\mathcal{H}}{\partial z^*}z^*+\frac{\partial\mathcal{H}}{\partial z}z\right)\boldsymbol{J}_z+\frac{\partial\mathcal{H}}{\partial z^*}\boldsymbol{J}_++\frac{\partial\mathcal{H}}{\partial z}\boldsymbol{J}_-\right). \tag{17}$$

2.3. Bilinear Hamiltonian

In this subsection we restrict ourselves to the study of the Hermitian bilinear Hamiltonian (for two degrees of freedom)

$$\boldsymbol{H}=\sum_i\alpha_i\boldsymbol{A}_i+\sum_j\beta_j\boldsymbol{B}_j+\sum_{i,j}\gamma_{i,j}\boldsymbol{A}_i\boldsymbol{B}_j, \tag{18}$$

where $i,j=0,\pm$, $\boldsymbol{A}_i$ and $\boldsymbol{B}_j$ are the semiclassical generators of either the harmonic oscillator or the rotating molecule: the index 0 is associated with the operator $\boldsymbol{a}^\dagger\boldsymbol{a}$ $(\boldsymbol{J}_z)$, the index $+$ with the operator $\boldsymbol{a}^\dagger$ $(\boldsymbol{J}_+)$ and the index $-$ with the operator $\boldsymbol{a}$ $(\boldsymbol{J}_-)$. This class of systems encompasses a rich variety of dynamical behavior ranging from the exactly solvable case of two linearly coupled harmonic oscillators to, e.g., the model we shall use for illustration which classical limit is chaotic. In this case, the classical Hamiltonian $\mathcal{H}(x,x^*,y,y^*)$ is the expectation value of the quantum Hamiltonian $\boldsymbol{H}$, in the product of coherent states $|x\rangle$, $|y\rangle$ for the first and the second degrees of freedom, respectively. It can be written in terms of the expectation values of the semiclassical generators, as $\mathcal{K}(\mathcal{A}_i,\mathcal{B}_j)$. By comparing the classical and semiclassical dynamics, as in the previous subsections, we end up with the SCH $\boldsymbol{H}_{\mathrm{sc}}$,

$$\boldsymbol{H}_{\mathrm{sc}}=\sum_i\alpha_i\boldsymbol{A}_i+\sum_j\beta_j\boldsymbol{B}_j+\sum_{i,j}\gamma_{i,j}(\mathcal{A}_i\boldsymbol{B}_j+\mathcal{B}_j\boldsymbol{A}_i), \tag{19}$$

which is, as in the previous cases of the form

$$\boldsymbol{H}_{\mathrm{sc}}=\sum_i\left(\frac{\partial\mathcal{K}}{\partial\mathcal{A}_i}\boldsymbol{A}_i+\frac{\partial\mathcal{K}}{\partial\mathcal{B}_i}\boldsymbol{B}_i\right). \tag{20}$$

In the particular case of two linearly coupled harmonic oscillators the SCD already provide for the exact solution.

3. Quantum corrections

The rest of this manuscript, in which we set $\hbar=1$, will be focused on bilinear Hamiltonian. For the application of the present formalism to point particle nonlinear Hamiltonian, see Ref. [9]. If we start with a coherent state $|x(0)\rangle\otimes|y(0)\rangle$, the SCD will yield at a later time t the state

$$\big|\psi(t)\big\rangle = \exp\big(i\eta(t)\big)\big|x(t)\big\rangle \otimes \big|y(t)\big\rangle, \tag{21}$$

where the coherent states $|x(t)\rangle$ and $|y(t)\rangle$ are the same as in the classical case, and the phase $\eta(t)$ is given by

$$\dot{\eta}(t) = \big\langle x(t), y(t)\big|\big(i\partial_t - \boldsymbol{H}_{\mathrm{sc}}(t)\big)\big|x(t), y(t)\big\rangle. \tag{22}$$

This type of phase, which correspond to a (generalized) action, is of course absent from the classical limit, but is crucial for the quantum corrections (see, for example, Eqs. (26) and (31)).

Now we turn to the corrections to the SCA. The exact Schrödinger equation for the whole system (with the tilde indicating the Schrödinger picture)

$$i\partial_t\big|\tilde{\psi}(t)\big\rangle = \big(\boldsymbol{H}_{\mathrm{sc}}(t) + \tilde{\boldsymbol{\Delta}}(t)\big)\big|\tilde{\psi}(t)\big\rangle,$$

where $\tilde{\boldsymbol{\Delta}}(t) = \boldsymbol{H} - \boldsymbol{H}_{\mathrm{sc}}(t)$, can be written in the SCA interaction picture (SCAIP) as (the absence of the tilde indicating the SCAIP)

$$\begin{aligned} i\partial_t\big|\psi(t)\big\rangle &= \boldsymbol{\Delta}(t)\big|\psi(t)\big\rangle \\ &= \sum_{i,j}\gamma_{i,j}\big(\boldsymbol{A}_i(t) - \mathcal{A}_i(t)\big) \\ &\qquad \times \big(\boldsymbol{B}_j(t) - \mathcal{B}_j(t)\big)\big|\psi(t)\big\rangle, \end{aligned} \tag{23}$$

up to time dependent terms that give rise to a global phase. Here the operators $\boldsymbol{A}_i(t)$, $\boldsymbol{B}_j(t)$ have evolved with the evolution operator $\bar{U}(t,0)$ of the SCA. Eq. (23) possesses a formal expansion

$$\begin{aligned} |\psi_t\rangle = \Bigg(1 - i\int_0^t dt_1\,\boldsymbol{\Delta}_{t_1} \\ - \int_0^t dt_1\int_0^{t_1} dt_2\,\boldsymbol{\Delta}(t_1)\boldsymbol{\Delta}(t_2) + \cdots\Bigg)|I\rangle, \end{aligned} \tag{24}$$

where $|I\rangle$ is the initial state $|x(0), y(0)\rangle$.

Let us see that all the terms in this expansion are readily written in terms of classical quantities. For example, up to terms of order Δ^2, the state of the system is a superposition of the initial state and the state $|D\rangle$, which sometimes is called a doorway state [10],

$$\big|\psi(t)\big\rangle \approx |I\rangle - iC(t)|D\rangle, \tag{25}$$

where $C(t)$ is a function to be defined below. In the present case the doorway state,

$$\begin{aligned} |D\rangle = N\big(1 - \big|x(0)\big\rangle\big\langle x(0)\big|\big) \\ \otimes \big(1 - \big|y(0)\big\rangle\big\langle y(0)\big|\big)\boldsymbol{H}\big|x(0)\big\rangle \otimes \big|y(0)\big\rangle, \end{aligned}$$

with N a normalization factor, is a *product* of generalized coherent states given by the action of the coherent displacement operator on new reference states: the Fock state with $n = 1$, in the case of point particles, and the state $|J, -J+1\rangle$ in the case of angular momenta. We introduce the notation $|x_1\rangle$ for these states, and $|x_2\rangle, |x_3\rangle, \ldots$ for the coherent states corresponding to the references states Fock state $n = 2$, $n = 3, \ldots$ or angular momentum along the z-axis $-J+2, -J+3, \ldots$.

The doorway amplitude in $|\psi(t)\rangle$ (Eq. (25)) is given by

$$\begin{aligned} C(t) &= \int_0^t dt_1\,c(t_1) \\ &= \int_0^t dt_1\,\sigma e^{i(S_{00}-S_{11})(t_1,0)} \\ &\quad \times \sum_{i,j}\gamma_{i,j}\,g^{\mathrm{A}}_{i+}\big(x(t_1)\big)g^{\mathrm{B}}_{j+}\big(y(t_1)\big), \end{aligned} \tag{26}$$

where $\sigma = \sqrt{\langle R_{\mathrm{A}}|\boldsymbol{A}_-\boldsymbol{A}_+|R_{\mathrm{A}}\rangle\langle R_{\mathrm{B}}|\boldsymbol{B}_-\boldsymbol{B}_+|R_{\mathrm{B}}\rangle}$, and $|R_{\mathrm{A}}\rangle$ ($|R_{\mathrm{B}}\rangle$) is the reference state of the degree of freedom A (B). It is evident that $C(t)$ is a function of classical trajectories only. The corresponding actions are $S_{00}(t,0) = \int_0^t d\tau\,\eta(\tau)$ and $S_{11}(t) = \int_0^t d\tau\,\langle x_1(\tau), y_1(\tau)|(i\partial_\tau - \boldsymbol{H}_{\mathrm{sc}}(\tau))|x_1(\tau), y_1(\tau)\rangle$. Observe that the subindices of the actions are related to the reference states of the degrees of freedom A and B.

From the expression of the first correction to the state, we immediately see that it already introduces all of the effects we avoided at the semiclassical level: superposition of states, nonlocality and entanglement.

In order to get the first order result we made use of the following identity which relays on the group structure of the semiclassical generators of the harmonic oscillator and the rotating molecule. In fact, it is this relation which enables one to express all order corrections in terms of classical trajectories and generalized coherent states

$$\boldsymbol{A}_i \mathcal{D}(x) = \mathcal{D}(x)\left(\sum_k g^{\mathrm{A}}_{i,k}(x)\boldsymbol{A}_k + k^{\mathrm{A}}_i(x)\right), \tag{27}$$

where $g_{i,k}(x)$ are functions of x specific to each one of the groups in question. Similar relations hold for the degree of freedom B. It is clear that the remaining corrections can also be written in terms of classical trajectories (and actions), but their quantum content will not be as transparent as in the leading correction. In fact, the second correction can be written as having a term proportional to the initial coherent state, but also terms proportional to generalized coherent states $|x_0, y_1\rangle$, $|x_1, y_0\rangle$, and $|x_2, y_2\rangle$, and their respective classical actions.

4. Sensitivity to initial conditions: a formal nonperturbative result

Classically one of the basic ingredients to define chaos is the high sensitivity to initial conditions. A formalization of this condition is heavily based on the notion of distance between trajectories. In establishing a quantum counterpart of this condition it is important to introduce a quantum measure of distance between states. A natural measure is given by the square modulus of the scalar product.[2] In what concerns our SCA, the squared modulus of the scalar product between different states of the manifold allows for a direct association of the distance between states with distances between phase space trajectories, since $|\langle z_1|z_2\rangle|^2$ is given by

$$\begin{cases} \exp\left(-|z_1 - z_2|^2\right) & \text{for point particles,} \\ \left(1 - \frac{|z_1-z_2|^2}{(1+|z_1|^2)(1+|z_2|^2)}\right)^{2J} & \text{for angular momenta.} \end{cases} \tag{28}$$

The important quantum tool which allows us to investigate the sensitivity to initial conditions of the quantum dynamics and eventually make connection to the well-known classical limit is the overlap between two time dependent states, which evolved from different initial conditions. It is well known that for unitary evolutions the scalar product is conserved in time. Observe that different initial states correspond to *different* SCAs, since the SCA is state dependent, due to self consistency. Thus the scalar product between two different initial states is to be written as

$$\begin{aligned} &\left\langle x'(0), y'(0)\middle| x(0), y(0)\right\rangle \\ &\quad = \left\langle x'(t), y'(t)\middle| x(t), y(t)\right\rangle \\ &\qquad + \left\langle x'(t), y'(t)\middle| \left(\delta'_{\mathrm{QC}}(t)\right)^\dagger \middle| x(t), y(t)\right\rangle \\ &\qquad + \left\langle x'(t), y'(t)\middle| \delta_{\mathrm{QC}}(t) \middle| x(t), y(t)\right\rangle \\ &\qquad + \left\langle x'(t), y'(t)\middle| \left(\delta'_{\mathrm{QC}}(t)\right)^\dagger \delta_{\mathrm{QC}}(t) \middle| x(t), y(t)\right\rangle, \end{aligned} \tag{29}$$

where we have written $U(t)$, the exact quantum time evolution operator as $\bar{U}^{(\prime)}(t)(1 + \delta^{(\prime)}_{\mathrm{QC}})$, with $\bar{U}^{(\prime)}(t)$ the SCA evolution operator corresponding to state $|x(t)^{(\prime)}, y(t)^{(\prime)}\rangle$ and $1 + \delta^{(\prime)}_{\mathrm{QC}}$ the evolution operator for the quantum corrections.

The first term on the rhs, $\langle x'(t), y'(t)|x(t), y(t)\rangle$ contains the SCA and is given by

$$\exp\left[-\frac{d(x(t) - x'(t))}{2} + i\Phi\left(x(t), x'(t)\right)\right] \\ \times \exp\left[-\frac{d(y(t) - y'(t))}{2} + i\Phi\left(y(t), y'(t)\right)\right],$$

where Φ is some phase which also depends on the classical trajectory and the corresponding group, and functions d can be determined by comparison with (28). This matrix element is proportional to the distance between the labels which, in the present case, corresponds precisely to the classical trajectories. Since the exact evolution preserves overlap, the sum of this term with the other three of Eq. (29), which contain the quantum corrections, should be conserved in time. Observe that the rate of change of this overlap can have two distinct origins, dephasing and/or change in magnitude. Both changes should be compensated by quantum corrections, but only the later one can be unambiguously connected to classical chaos, since, if the system is classically chaotic this distance will exponentially grow and, as a consequence, the overlap involving only the SCA will decrease accordingly. This is a quantum counterpart of the fact that classically chaotic systems exhibit high sensitivity to initial conditions. The corresponding quantum system

[2] Some proposals have been made related to scalar product of wavefunction evolved from different Hamiltonian (not different wavefunctions). See A. Peres, Phys. Rev. A 30 (1984) 1610; R.A. Jalabert, H.M. Patawski, Phys. Rev. Lett. 86 (2001) 2490.

will exhibit a high sensitivity to the initial state in what concerns the production rate of nonunitary quantum corrections to the magnitude of the overlap. However, other observables will have different, independent, time scales, sometimes much shorter, as, for example, that for the entanglement process. Consequently, entanglement is not always a good measure of classical chaotic behavior.

In order to characterize the degree of entanglement we will calculate the idempotency defect (or linear entropy) $\delta(t) = 1 - \mathrm{Tr}_A(\rho_A(t))^2$, where the reduced density $\rho_A(t)$ is given by $\rho_A(t) = \mathrm{Tr}_B |\psi(t)\rangle\langle\psi(t)|$. Using the expansion (24) up to second order, writing $|I\rangle = |I_A\rangle \otimes |I_B\rangle$, and calculating the idempotency defect in the SCAIP we obtain to second order

$$\delta(t) = 4\,\mathrm{Re}\int_0^t dt_1 \int_0^{t_1} dt_2\, c^*(t_1)c(t_2). \tag{30}$$

It should be emphasized that the time scale for nonunitary quantum corrections to the overlap is the one which is essentially linked to the Lyapunov exponents. Should the exponential separation occur at early times, a significant increase in linear entropy will be noticed. In case the two time scales are very different, this effect, although present, will be rendered less conspicuous by the time development of quantum correlations stemming from the other sources. This will become clear in the example below.

5. Application to the classically chaotic maser model

The classically chaotic maser Hamiltonian

$$H = \epsilon \boldsymbol{J}_z + \omega \boldsymbol{a}^\dagger \boldsymbol{a} + \frac{G}{\sqrt{J}}\big(\boldsymbol{a}^\dagger \boldsymbol{J}_- + \boldsymbol{a}\boldsymbol{J}_+\big) + \frac{G'}{\sqrt{J}}\big(\boldsymbol{a}^\dagger \boldsymbol{J}_+ + \boldsymbol{a}\boldsymbol{J}_-\big)$$

belongs to the class of bilinear Hamiltonian (18). We assume that the field (the atom) corresponds to the degree of freedom A (B). Field coherent states are characterized by $x(t)$ while spin coherent states by $y(t)$. The SCH is of form (20) with $\boldsymbol{A}_{-(+)} = \boldsymbol{a}^{(\dagger)}$, $\boldsymbol{A}_0 = \boldsymbol{a}^\dagger\boldsymbol{a}$, $\boldsymbol{B}_\pm = \boldsymbol{J}_\pm$, $\boldsymbol{B}_0 = \boldsymbol{J}_z$, and

$$\frac{\partial\mathcal{K}}{\partial z^*} = -i2\sqrt{J}\,\frac{Gy + G'y^*}{1 + y^*y},$$
$$\frac{\partial\mathcal{K}}{\partial z} = \left(\frac{\partial\mathcal{K}}{\partial z^*}\right)^*\ \frac{\partial\mathcal{K}}{\partial |z|^2} = \omega,$$
$$\frac{\partial\mathcal{K}}{\partial \mathcal{J}_+} = -i\,\frac{Gx + G'x^*}{\sqrt{J}},$$
$$\frac{\partial\mathcal{K}}{\partial \mathcal{J}_-} = \left(\frac{\partial\mathcal{K}}{\partial \mathcal{J}_+}\right)^*\ \frac{\partial\mathcal{K}}{\partial \mathcal{J}_z} = \epsilon.$$

As explained in the preceding sections this Hamiltonian induces a classical-like dynamics. The nonlinearity of these equations arise from the self consistency of SCA. If the labels are scaled as $x = \sqrt{4J}X$ for the field, $y = Y$ for the atom, and time as $t = t_c/(4J)$, then the equations for $X(t_c)$ and $Y(t_c)$ will become independent of J and correspond to the classical limit of Heisenberg's equations of motion.

For this model the perturbation $\Delta(t)$ in the SCAIP is given by

$$\Delta(t) = \frac{1}{\sqrt{J}}\big(\boldsymbol{a}(t) - x(t)\big)\bigg(G\boldsymbol{J}_+(t) + G'\boldsymbol{J}_-(t) - \frac{2J(Gy^*(t) + G'y(t))}{1 + |y|^2(t)}\bigg) + \text{h.c.},$$

where the time dependence of the operators is the one acquired in the semiclassical evolution. Employing the expansion (24) to first order, it is a simple matter to give an analytic expression for $c(t)$, the key ingredient for evaluation of both the first order correction for the state, Eqs. (25) and (26), and the idempotency defect, Eq. (30). We have

$$c(t) = \frac{\sqrt{2}e^{i(S_{00} - S_{11})(t)}}{1 + |y(t)|^2}\big(G' - Gy^2(t)\big), \tag{31}$$

where S_{00} (S_{11}) is the generalized action for the coherent state with fiducial state $|n = 0\rangle \otimes |J, -J\rangle$ ($|n = 1\rangle \otimes |J, -J + 1\rangle$). In this case the doorway state $|D\rangle$ is given by $|D\rangle = |x_1, y_1\rangle$.

The time development, in the SCA, of the magnitude of the overlap between (a) two product coherent states initially centered in the classically chaotic phase space region and (b) two product coherent states initially centered in the regular region are shown in Fig. 1. It illustrates the dramatic effect of the classically chaotic motion on quantum corrections to the

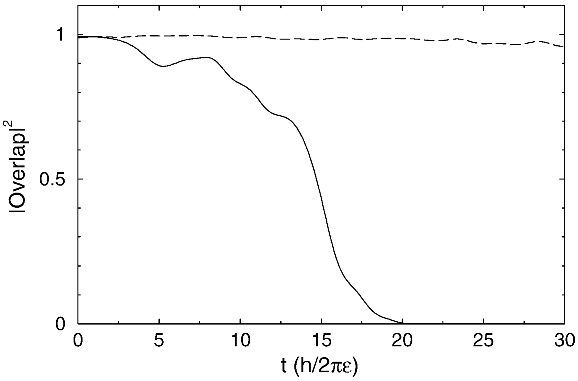

Fig. 1. Squared modulus of the overlap between two neighboring states $|\langle x_1(t), y_1(t)|x_2(t), y_2(t)\rangle|^2$ in the SCA for mean energy $E = 8.5$, $J = 9/2$ in a resonant ($\epsilon = 1 = \omega$) nonintegrable case with $G = 0.5$, $G' = 0.2$ for conditions in the chaotic region (continuous line) and regular region (dashed line). Fig. 1 of Ref. [7] portrays the corresponding Poincaré section for the spin degree of freedom (notice that a different parametrization has been used). Chaotic initial conditions $(x_1, y_1) = (5.7263433, -0.24253563)$, $(x_2, y_2) = (5.7778567, -0.26845243)$. Regular initial conditions $(x_1, y_1) = (3.615516, 0.53452248)$, $(x_2, y_2) = (3.68977334, 0.50086791)$.

overlap. In fact, any substantial drops of the overlap are matched by corresponding divergences of the classical trajectories. Therefore the magnitude of the SCA overlap is mainly controlled by classical dynamics. Of course, this needs not hold for other quantum observables. Entanglement, for example, is a quantum property with a smaller time scale. The expected increases in this quantity at the times when the magnitude of the overlap diminishes are in fact rendered invisible due to previous near saturation of our measure of the entanglement by interaction effects having negligible impact on this quantity (see Ref. [7]).

6. Conclusions and remarks

We have found a *semiclassical approximation* for arbitrary systems composed of point particles and angular momenta, with Hamiltonian which admit a Taylor expansion in the semiclassical generators for the harmonic oscillator and the rotating molecule, for initial coherent states, which satisfy the following requirements: labels with classical physical meaning, no superposition principle, minimum quantum nonlocality effects, and (quantum) semiclassical dynamics which coincide with the Hamiltonian equations for labels. The coincidence between semiclassical and classical dynamics allows for an extension of the concept of divergence of classical trajectories to the quantum realm, using the overlap between initial states centered on neighboring points of the phase space. Of course, the possibility of mimicking the classical sensitivity to initial conditions rests on the nonlinear character of the semiclassical Hamiltonian, which is a function of the quantum state.

This approximation allows for a perturbative expansion around it, which can be written in terms of classical ingredients: trajectories and (generalized) actions. Since the state properties and the averages properties of the semiclassical generators can be independently addressed, this formalism can be used to show that the characteristic times for the appearance of quantum effects for the state and for the averages are completely different [9].

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References

[1] W.H. Zurek, Phys. Rev. D 24 (1981) 1516;
For recent reviews J.M. Raimond, et al., Rev. Mod. Phys. 73 (2001) 565;
A. Zeilinger, Rev. Mod. Phys. 71 (1999) S281, and references therein.
[2] O. Bohigas, et al., Phys. Rev. Lett. 52 (1984) 1.
[3] M.C. Gutzwiller, J. Math. Phys. 12 (1971) 343.
[4] G.P. Berman, G.M. Zaslavsky, Physica A 91 (1978) 450;
G.P. Berman, A.M. Iomin, G.M. Zaslavsky, Physica D 4 (1981) 113;
G.P. Berman, A.M. Iomin, Theor. Math. Phys. 77 (1988) 1097.
[5] G.P. Berman, E.N. Bulgakov, G.M. Zaslavsky, Chaos 2 (1992) 257;
G.P. Berman, E.N. Bulgakov, D.D. Holm, V.I. Tsifrinovich, Phys. Lett. A 181 (1993) 296;

G.P. Berman, E.N. Bulgakov, D.D. Holm, Phys. Rev. A 49 (1994) 4943.
[6] W.H. Zurek, J.P. Paz, Physica D 83 (1995) 300.
[7] K. Furuya, et al., Phys. Rev. Lett. 80 (1998) 5524.
[8] P.A. Miller, S. Sarkar, Phys. Rev. A 60 (1999) 1542.
[9] A.C. Oliveira, M.C. Nemes, K.M. Fonseca Romero, Phys. Rev. E 68 (2003) 036214.
[10] N. Zagury, A.F.R. de Toledo Piza, Phys. Rev. A 50 (1994) 2908.