phys. stat. sol. (b) **220**, 799 (2000)

Subject classification: 71.23.–k

Non-Markovian Effects in a Simple Photonic Band Gap Cavity

K. M. FONSECA-ROMERO[1])

Departamento de Física, Universidad de los Andes, A.A. 4976, Bogotá, Columbia

(Received November 1, 1999)

We propose a simple model for a cavity in a 1D photonic band gap material. It is shown that the exact dynamics contains three real functions of time, implicitly defined through an integrodifferential equation, all of them experimentally accessible. Several initial conditions are analyzed: (generalized) coherent states at finite temperature and Schrödinger cat states at zero temperature. We use the simplest energy momentum relation which allows for a band gap and show that the perturbative solution of the integrodifferential equation presents non-Markovian features.

Recently great attention has been paid to the so-called photonic band gap (PBG) materials [1] and the consequent new electromagnetic effects and devices. Here we propose a very simple model for an imperfect cavity (a Fabry-Pérot resonator) imbedded in a PBG material. We solve the model and show that the modified dispersion relation for light leads to non-Markovian effects. Moreover, we show that the necessary information to obtain the dynamical state of the cavity field is encoded into three real functions of time. These functions, which can be measured, are the average photon number and two orthogonal quadratures. We assume the usual Hamiltonian for leaky cavities

$$H = \hbar\omega(a^\dagger a + 1/2) + \hbar \sum_k \omega_k(a_k^\dagger a_k + 1/2) + \hbar' \sum_k g_k(a^\dagger a_k + a_k^\dagger a)\,, \tag{1}$$

where ω is the frequency of the ideal cavity, ω_k are the frequencies of the PBG material, and a, $a^\dagger$ and a_k, $a_k^\dagger$ are the respective annihilation and creation operators. We assume that the g_k coefficients are all equal. We trace the evolution equation for the Wigner function (as in Refs. [2, 3]) and obtain the following equation for the reduced density operator of the system [4]

$$\begin{aligned}\frac{\mathrm{d}\varrho}{\mathrm{d}t} &= \frac{1}{i\hbar}\,[\hbar(\omega+\delta)\,a^\dagger a,\,\varrho] + (\lambda+\varepsilon)\,(2a\,\bullet\,a^\dagger - a^\dagger a\,\bullet - \bullet\,a^\dagger a)\,\varrho\\ &\quad + \varepsilon(2a^\dagger\,\bullet\,a - aa^\dagger\,\bullet - \bullet\,aa^\dagger)\,\varrho = \mathcal{L}(t)\,\varrho(t)\,,\end{aligned} \tag{2}$$

where the usual dot superoperator convention has been used. The usual Born-Markov RWA master equation is of this form with constant coefficients [6]. The real functions $\delta(t)$, $\lambda(t)$, $\varepsilon(t)$ which appear in (2) can be determined as follows. The logarithm of the solution of the integrodifferential equation

$$\dot{\alpha} + i\omega_0\alpha + \int_0^t \mathrm{d}\tau \sum_k g_k^2\,\mathrm{e}^{-i\omega_k(t-\tau)}\,\alpha(\tau) = 0\,, \tag{3}$$

[1]) e-mail: kfonseca@uniandes.edu.co

subject to the initial condition $\alpha(0) = 1$, gives $-i\Omega - \Lambda$, with

$$\Omega(t) = \int_0^t \mathrm{d}\tau(\omega + \delta)(\tau), \qquad \Lambda(t) = \int_0^t \mathrm{d}\tau\, \lambda(\tau). \tag{4}$$

Finding $\varepsilon(t)$ is somewhat more cumbersome. After some algebra we arrive at

$$\varepsilon(t) = \frac{\exp(-2\Lambda(t))}{2} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(2\Lambda(t)) \sum_k \frac{c_k^2}{\exp\left(\frac{\hbar\omega_k}{k_\mathrm{B}T}\right) - 1} \left|\int_0^t \mathrm{d}\tau \exp(i\omega_k\tau - i\Omega(\tau) - \Lambda(\tau))\right|^2\right). \tag{5}$$

These results can be compared to those obtained in Ref. [5].

Some results can be obtained at once from (2): premultiplying by a and taking the trace we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\alpha\rangle = \frac{\mathrm{d}\alpha}{\mathrm{d}t} = (-i(\omega + \delta) - \lambda)\,\alpha, \tag{6}$$

which can be immediately solved to give

$$\alpha(t) = \exp(-i\Omega(t) - \Lambda(t))\,\alpha(0). \tag{7}$$

Notice that this result is *independent* of ε, i.e., it does not depend on the temperature. Premultiplying (2) by $a^\dagger a$, and taking the trace we have the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a^\dagger a\rangle(t) = -2\lambda\langle a^\dagger a\rangle(t) + 2\varepsilon, \tag{8}$$

with the solution

$$\begin{aligned} \langle a^\dagger a\rangle(t) &= \exp(-2\Lambda(t))\langle a^\dagger a\rangle(0) + \mathcal{N}(t), \\ \mathcal{N}(t) &= 2\,\mathrm{e}^{-2\Lambda(t)} \int_0^t \mathrm{d}\tau\, \varepsilon(\tau)\,\mathrm{e}^{2\Lambda(\tau)}, \end{aligned} \tag{9}$$

where it is evident that $\mathcal{N}(t)$ vanishes in the limit of zero temperature.

We can use Lie algebraic methods [7] to find the evolution superoperator $\mathcal{U}$. Let us observe that the superoperators $\mathcal{M}$, $\mathcal{P}$, $\mathcal{J}$, $\mathcal{R}$, $\mathcal{I}$, given by

$$\begin{aligned} &\mathcal{M} = a^\dagger a\,\bullet, \qquad \mathcal{P} = \bullet\, a^\dagger a, \qquad \mathcal{J} = a \bullet a^\dagger, \\ &\mathcal{R} = a^\dagger \bullet a, \quad \text{and} \quad \mathcal{I} = 1\bullet = \bullet 1 \end{aligned} \tag{10}$$

do form a Lie algebra under commutation. This fact can be exploited: it is well known that in this case the evolution superoperator can be written as

$$\mathcal{U}(t) = A(t)\,\mathrm{e}^{u(t)\mathcal{R}}\,\mathrm{e}^{x(t)\mathcal{M}}\,\mathrm{e}^{y(t)\mathcal{P}}\,\mathrm{e}^{z(t)\mathcal{J}}. \tag{11}$$

A somewhat lengthy but straightforward calculation leads to $A(t) = (1 + \mathcal{N}(t))^{-1}$, $u(t) = A(t)\,\mathcal{N}(t)$, $x(t) = -\frac{1}{2}\ln[1 + \mathcal{N}(t)] + \Lambda(t) + i\Omega(t) = y^*(t)$, $z(t) = 1 - A(t)$.

It is not difficult to prove that one of the characteristics of Eq. (2) is that if $\varrho(t)$ solves it, then $U^\dagger \varrho U$ with $U = \exp(\sigma a^\dagger - \sigma^* a) = D(\sigma)$, the displacement operator, and

$$\sigma(t) = \sigma(0) \exp(-i\Omega(t) - \Lambda(t)), \qquad \sigma^*(t) = (\sigma(t))^*, \tag{12}$$

is also a solution of (2).

We now turn to the evaluation of the density matrix evolved with the evolution superoperator found above. We chose initial states that are relevant from the point of view of quantum optics. We begin with the ground state, $\varrho(0) = |0\rangle\langle 0|$. We have

$$\varrho(t) = \mathcal{U}(t)\, |0\rangle\langle 0| = \sum_{n=0}^{\infty} \frac{1}{1+\mathcal{N}(t)} \left(\frac{\mathcal{N}(t)}{1+\mathcal{N}(t)}\right)^n |n\rangle\langle n| = \sum_{n=0}^{\infty} P_n\, |n\rangle\langle n|\,. \tag{13}$$

The above formula exemplifies what is called the decomposition in natural orbitals in such a way that the quantities P_n can be directly interpreted as probabilities [8].

For an initial Fock state we see that the density matrix for positive times is given by

$$\mathcal{U}(t)\, |m\rangle\langle m| = \sum_{s=0}^{\infty} P_{m,s}(t)\, |s\rangle\langle s| \tag{14}$$

with

$$P_{m,s}(t) = \frac{e^{-2m\Lambda(t)}}{(1+\mathcal{N}(t))^{m+1}} \frac{m!}{s!} \sum_{k=0}^{\min(m,s)} \frac{([1+\mathcal{N}(t)]\, e^{2\Lambda(t)} - 1)^k}{(m-k)!\,(s-k)!} \left(\frac{\mathcal{N}(t)}{1+\mathcal{N}(t)}\right)^{s-k}. \tag{15}$$

The above density operator has also been expressed in terms of its natural orbits, therefore, the quantities $P_{m,s}(t)$ can be interpreted as probabilities. The transformation property discussed above allows us to write the evolution of an initial generalized coherent state $|\sigma_0 m\rangle = D(\sigma_0)\,|m\rangle$. We have

$$\mathcal{U}(t)\, |\sigma_0 m\rangle\langle \sigma_0 m| = \sum_{s=0}^{\infty} P_{m,s}(t)\, |\sigma(t)\, s\rangle\langle \sigma(t)\, s|\,, \tag{16}$$

with $P_{m,s}(t)$ as above, and $\sigma(t)$ given by (12).

At zero temperature things are simpler. For example, the evolution superoperator now reads

$$\mathcal{U}(t) = e^{x\mathcal{M}}\, e^{y\mathcal{P}}\, e^{z\mathcal{J}}\,, \tag{17}$$

with

$$x(t) = -i\Omega(t) - \Lambda(t)\,, \qquad y(t) = x^*(t)\,, \qquad z(t) = 1 - \exp(-2\Lambda(t))\,. \tag{18}$$

Applying this evolution operator (17) to an initial density matrix element $|\sigma_0\rangle\langle\sigma_0'|$, one obtains

$$\mathcal{U}(t)\, |\sigma_0\rangle\langle\sigma_0'| = \frac{\langle\sigma_0' \mid \sigma_0\rangle}{\langle\sigma'(t) \mid \sigma(t)\rangle}\, |\sigma(t)\rangle\langle\sigma'(t)|\,. \tag{19}$$

The evolution of an initial even ($\varrho_{\sigma_0\,\mathrm{e}}$) or odd cat ($\varrho_{\sigma_0\,\mathrm{o}}$) state can be calculated from (19). Indeed,

$$\varrho_{\sigma_0\,\mathrm{e(o)}} = N_{\mathrm{e(o)}}(\sigma_0)\, (|\sigma_0\rangle,\, |-\sigma_0\rangle) \begin{pmatrix} 1 & (-)1 \\ (-)1 & 1 \end{pmatrix} \begin{pmatrix} \langle\sigma_0| \\ \langle-\sigma_0| \end{pmatrix}, \tag{20}$$

where $N_{e(o)}(\sigma_0) = (1 + (-)\langle -\sigma_0 \mid \sigma_0\rangle)^{-1}/2$ is a normalization factor, evolves as follows:

$$\mathcal{U}\varrho_{\sigma_0\,e(o)} = N_{e(o)}(\sigma_0)\,(|\sigma_t\rangle,\, |-\sigma_t\rangle) \times \begin{pmatrix} 1 & \dfrac{(-)\langle -\sigma_0 \mid \sigma_0\rangle}{\langle -\sigma(t) \mid \sigma(t)\rangle} \\ \dfrac{(-)\langle -\sigma_0 \mid \sigma_0\rangle}{\langle -\sigma(t) \mid \sigma(t)\rangle} & 1 \end{pmatrix} \begin{pmatrix} |\sigma_t\rangle \\ |-\sigma_t\rangle \end{pmatrix}. \tag{21}$$

We can rewrite Eq. (21) in a more convenient way, in terms of natural orbitals, as

$$\mathcal{U}(t)\,\varrho_{\sigma_0\,e(o)} = p^{e(o)}_{e(o)}(t)\,\varrho_{\sigma(t)\,e(o)} + p^{o(e)}_{e(o)}(t)\,\varrho_{\sigma(t)\,o(e)}\,, \tag{22}$$

with

$$p^{e(o)}_{e(o)}(t) = \frac{1}{2}\,\frac{1 + (-)\langle -\sigma(t) \mid \sigma(t)\rangle}{1 + (-)\langle -\sigma_0 \mid \sigma_0\rangle}\left(1 + \frac{\langle -\sigma_0 \mid \sigma_0\rangle}{\langle -\sigma(t) \mid \sigma(t)\rangle}\right), \tag{23}$$

$$p^{o(e)}_{e(o)}(t) = \frac{1}{2}\,\frac{1 - (-)\langle -\sigma(t) \mid \sigma(t)\rangle}{1 + (-)\langle -\sigma_0 \mid \sigma_0\rangle}\left(1 - \frac{\langle -\sigma_0 \mid \sigma_0\rangle}{\langle -\sigma(t) \mid \sigma(t)\rangle}\right). \tag{24}$$

Observe that an initial cat state evolves as a mixture of even and odd cat states. Equation (23) gives the probability of the cat state of the same parity as the initial state, and Eq. (24) the probability of the cat state of the other parity. This example is both of theoretical and practical importance because this kind of states, which probe the classical-quantum frontier, are currently being prepared in some laboratories [9].

We have seen that Ω and Λ are all we need to characterize the effect of the bath on the main oscillator. A measurement of the number of photons, whether using the traditional counting of photons or the recently proposed non-demolition measurement [10], leads to the determination of the function $\Lambda(t)$. The experimental set-up to measure the Wigner function [11] can be used to determine experimentally both $\Lambda(t)$ and $\Omega(t)$. The subsequent observation of the mean energy and a fitting procedure lead to the last unknown function $\varepsilon(t)$.

We show next that when a cavity has been embedded into a PBG material then there are non-Markovian effects, that is, the functions λ, δ and ε are not constants. Thus, the usually employed Markovian approximation is no longer valid. We use the simplest possible model for a PBG. We assume the following dispersion relation:

$$\omega(k) = \begin{cases} c_1 k & \text{if} \quad k < k_0\,, \\ c_2 k & \text{if} \quad k_0 < k < k_D\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{25}$$

where k_0 is the value of the wave number for which the discontinuity occurs, k_D is the Debye wave number and $c_1 < c_2$ are constants. Notice that if $c_1 = c_2 = c$, $\mu = g^2 k_D/(c^2 k_0(k_D - k_0)) \ll 1$ and $k_D \ll k_0$ we have $-i\Omega(t) - \Lambda(t) = \mu - g^2\pi t/c - i(\omega_0 - \log(k_D/k_0 - 1))\,t$. The constant term is known as the initial slip. This result corresponds to the expected Markovian behavior. Now, if $c_2 = c$, $c_1 = (1-x)\,c/(1-x)$, and

$k_0 = (1+x)\,\omega_0/c$, in such a way that the middle of the band gap coincides with the natural frequency of the cavity, and setting $z\omega_0 = ck_{\mathrm{D}} - \omega_0$ and $\tau = \omega_0 t$ we obtain

$$-\Lambda = \frac{g^2\omega_0}{c}\left(\tau\left(\mathrm{Si}\,(z\tau) + \frac{(1-x)\,\mathrm{Si}\,(\tau)}{1-x} - \frac{2x\,\mathrm{Si}\,(x\tau)}{1-x}\right)\right.$$
$$\left. - \frac{1-\cos\,(z\tau)}{z} - \frac{(1+x)\,(1-\cos\,(\tau))}{1-x} + \frac{2-2\cos\,(x\tau)}{1-x}\right), \tag{26}$$

$$\Omega - \tau = \frac{g^2\omega_0}{c}\left(\tau\left(-\log\,(z) + \mathrm{Ci}\,(z\tau) - \frac{(1+x)\,\mathrm{Ci}\,(\tau) + 2x\,\mathrm{Ci}\,(x\tau) - 2x\log\,(x)}{1-x}\right)\right.$$
$$\left. - \frac{\sin\,(z\tau)}{z} + \frac{(1+x)\sin\,(\tau) - 2\sin\,(x\tau)}{1-x}\right). \tag{27}$$

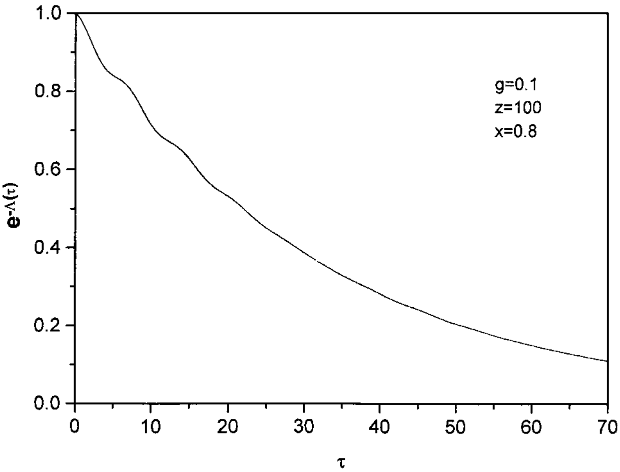

Fig. 1. Energy as function of time

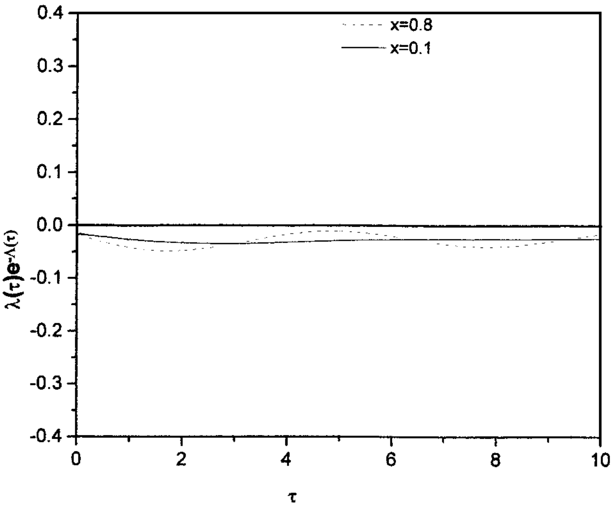

Fig. 2. Energy derivative

The notation Ci, Si corresponds to the integral cosine and sine functions. Figure 1 shows the behavior of the energy as a function of time. We have chosen $g = 0.1$, $c = 0.5$, $z = 100$, $x = 0.8$. Observe the ripples in the graphic: they signal the non-Markovian behavior. This behavior can be traced back to the interference of the x-dependent terms. In particular, it can be shown that there are two terms which produce an attenuated beat for x close to one. In Fig. 2 we plot the energy derivative. Observe that non-Markovian behavior can be inferred only for $x = 0.8$ and not for $x = 0.1$. The non-Markovian signal can also be wiped out for energy measurements with large error.

In conclusion, we have shown that if the usual Hamiltonian for leaky cavities is adequate for PBG environments, large photonic bandgaps around the cavity frequency produce non-Markovian effects. In particular, the cavity energy, an experimental accessible quantity, clearly displays these effects.

Acknowledgement This work was partly funded by COLCIENCIAS.

References

[1] E. Yablonovitch, Phys. Rev. Lett. **58**, 2059 (1987).
J. D. Joannopoulos, R. D. Meade, and J. N. Winn, Photonic Crystals: Molding the Flow of Light, Princeton University Press, Singapore 1995.
[2] J. Anglin and S. Habib, Mod. Phys. Lett. A **11**, 2655 (1996).
[3] J. J. Halliwell and T. Yu, Phys. Rev. D **53**, 2012 (1996).
[4] K. M. Fonseca-Romero and M. C. Nemes, unpublished.
[5] J. J. Hope, Phys. Rev. A **55**, 2531 (1997).
[6] W. H. Louisell, Quantum Statistical Properties of Radiation, Wiley, New York 1973.
[7] H. E. Borges, Ph. D. Thesis, Universidade Federal de Minas Gerais, 1996.
[8] M. C. Nemes and A. F. R. de Toledo Piza, Physica **137A**, 367 (1986).
[9] M. Brune, S. Haroche, V. Lefevre, J. M. Raimond, and N. Zagury, Phys. Rev. Lett. **65**, 976 (1990).
M. Brune, S. Haroche, J. M. Raimond, L. Davidovich, and N. Zagury, Phys. Rev. A **45**, 5193 (1992).
[10] M. Brune, E. Hagley, J. Dreyer, X. Maître, A. Maali, C. Wunderlich, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **77**, 4887 (1996).
[11] L. G. Lutterbach and L. Davidovich, Phys. Rev. Lett. **78**, 2547 (1997).